$\mathcal{G}^{+++}$ and Brane Solutions

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Abstract

We demonstrate that the very extended $\mathcal{G}^{+++}$ group element of the form $g_A = \exp(-\frac{1}{(\beta,\beta)} \ln N \beta \cdot H) \exp((1-N)E_\beta)$ describes the usual BPS, electric, single brane solutions found in $\mathcal{G}^{+++}$ theories.

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1 Introduction

Very little is known about M theory itself, but investigations of the associated supergravity theories are beginning to reveal some of its underlying symmetry. Study of the properties of the eleven dimensional supergravity theory has lead to the recent conjecture [1] that M theory possesses an E_{11} Kac-Moody symmetry. Indeed it was argued [1] that an extension of eleven dimensional supergravity should possess an E_{11} symmetry that was non-linearly realised. A benefit of this approach is that the symmetries found when the eleven dimensional supergravity theory was dimensionally reduced are natural occurrences of there being an E_{11} symmetry in the eleven dimensional theory.

The same analysis was applied to the IIA and IIB supergravity theories which were conjectured to be part of a larger theory which also possessed an E_{11} symmetry [1, 2]. This is consistent with the idea that the type II string theories in ten dimensions and an eleven dimensional theory are part of a single theory called M theory. Indeed their common E_{11} origin provides an explicit relation between the type IIB theory in ten dimensions and the eleven dimensional theory which are not related by dimensional reduction [3].

Arguments similar to those advocated for eleven dimensional supergravity in [1] were proposed to apply to gravity [4] in D dimensions, the effective action of the closed bosonic string [1] generalised to D dimensions and the type I supergravity theory [5] and the underlying Kac-Moody algebras were identified. It was realised that the algebras that arose in all these theories were of a special kind and were called very extended Kac-Moody algebras [6]. Indeed, for any finite dimensional semi-simple Lie algebra $\mathcal{G}$ one can systematically extend its Dynkin diagram by adding three more nodes to obtain an indefinite Kac-Moody algebra denoted $\mathcal{G}^{+++}$. In this notation E_{11} is written as E_8^{+++}. The algebras for gravity, the closed bosonic string, and type I supergravity being A_{D-3}^{+++} [4], D_{D-2}^{+++} [1] and D_8^{+++} [5] respectively. The Dynkin diagrams for the very extended algebras considered in this paper are given in Appendix B.

It was proposed in [1, 4, 5, 6] and [7, 8, 9, 10], that the non-linear realisation of any very extended algebra $\mathcal{G}^{+++}$ leads to a theory, called $\mathcal{V}_{\mathcal{G}}$ in [9], that at low levels includes gravity and the other fields and it was hoped that this non-linear realisation contains an infinite number of propagating fields that ensures its consistency. Indeed, it has been shown [9] that the non-linear realisation of any of the very extended algebras $\mathcal{G}^{+++}$ leads to a theory that contains gravity, gauge fields of various rank and possibly a dilaton at low levels.

If one dimensionally reduces a supergravity theory on a torus to $D = 3$ one obtains a generic theory containing gravity, gauge fields of various rank and possibly a dilaton that is described by the properties of its scalars. A non-trivial field strength in three dimensions may have at most two indices making it the dual of the field strength derived from the scalar. In some cases these scalars belong to a non-linear realisation of some finite dimensional Lie group, $\mathcal{G}$. A set of theories whose dimensional reductions to three dimensions yielded every possible $\mathcal{G}$ was found in [11]. The maximally oxidised theories corresponding to $\mathcal{G}$ are those in the highest dimension, D, such that upon dimensional reduction

to three dimensions one finds the scalars belonging to the non-linear realisation $\mathcal{G}$ [11]; for a detailed discussion of oxidation in the context of group theory the reader is referred to [12]. Reference [4] considered the dimensional reduction of certain theories containing gravity, gauge fields of various rank and a dilaton and found that only for very specific couplings did the scalars belong to a non-linear realisation of $\mathcal{G}$ and in all cases these couplings were oxidised theories. Indeed a correspondence between the low-level fields in the adjoint representation of $\mathcal{G}^{+++}$ and the maximally oxidised theory associated with $\mathcal{G}$ has been demonstrated in reference [9].

There is a very substantial literature on brane solutions in supergravity and, indeed, in a generic theory consisting of gravity, gauge fields of various rank and a dilaton we will find that our equation (1) encodes the electric brane solutions found in these theories [13, 14]. The solutions for the oxidised theories mentioned above were considered in [8].

It was shown in [3] that the usual BPS solutions of the type IIA and type IIB ten dimensional supergravity theory and the eleven dimensional theory possess a general formulation in terms of E_{11} group elements, namely, $\exp(-\frac{1}{2}\ln N\beta \cdot H)\exp((1 - N)E_\beta)$. The form of this expression makes it a simple process to write down any solution from a particular E_{11} group element, and vice-versa.

In this paper we generalise the result of [3] so that it may be applied to any $\mathcal{G}^{+++}$ algebra. We find that the group element takes the general form

$$g_A = \exp(-\frac{1}{(\beta,\beta)}\ln N\beta \cdot H)\exp((1 - N)E_\beta) \tag{1}$$

Where (β,β) is the norm squared of the root whose corresponding generator is E_β, when the squared length of the roots on the gravity line have been normalised to two. For the cases where $(\beta,\beta) = 2$ the group element becomes that given in [3]. N is a harmonic function taking the form

$$N = 1 + \frac{k}{r^{[(D-2)-2\sum\limits_{i\in GL}i\sum\limits_{j\notin GL}\rho_{ij}(\alpha_j,\beta) - \sum\limits_{j\notin GL}D]}} \tag{2}$$

Where GL refers to nodes of the Dynkin diagram found on the gravity line of each theory, $\rho_{ij} = \frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)}$ is the metric of the Cartan subalgebra, D is the dimension of spacetime and $r^2 = \delta^{ab}x_bx^a$ is the radial extension in the transverse coordinates, x^a.

In section 2 we demonstrate the use of equation (1) for the simply-laced group D_{n-3}^{+++} and deduce the usual BPS solutions, while in section 3 we apply the method to the non-simply laced groups B_{n-3}^{+++} and F_4^{+++}. Each of the $\mathcal{G}^{+++}$ theories we consider in this paper has a pp-wave solution arising from the gravity sector but their derivation does not differ significantly from those given in [3] and are not calculated in this paper. Finally, in section 4 we consider some of the consequences of this result in relation to brane solutions emerging from the higher order content of the $\mathcal{G}^{+++}$ theories. In Appendix A we give

2

the application of the solution to the remaining very extended groups, with the exception of C_{n-3}^{+++1}, and the relevant very extended Dynkin diagrams are given in Appendix B.

1.1 Kac-Moody Algebras

A Kac-Moody algebra is constructed from its Cartan matrix, A_{ab}[2] and the Chevalley generators. A Cartan matrix A_{ab} is given in terms of the simple roots, α_a, of an algebra as

$$A_{ab} = 2\frac{(\alpha_a, \alpha_b)}{(\alpha_a, \alpha_a)} \tag{3}$$

Where (α_a, α_b) is the Kac-Moody generalised Killing form acting on the corresponding elements of the Cartan sub-algebra. The diagonal elements of A_{ab} are all 2 and the off-diagonal elements are negative integers, with the zeroes being symmetrically distributed such that

$$A_{ab} = 0 \qquad \Leftrightarrow \qquad A_{ba} = 0 \tag{4}$$

The generators associated with positive simple roots are labeled $E_a = E_{\alpha_a}$ and those associated to negative simple roots $F_a \equiv E_{-\alpha_a}$. A Kac-Moody algebra is then uniquely constructed from the commutators of E_a, F_a and the Cartan sub-algebra generators, H_a, subject to the following Serre relations

$$[H_a, H_b] = 0$$
$$[H_a, E_b] = A_{ab}E_b \qquad\qquad [H_a, F_b] = -A_{ab}F_b \tag{5}$$
$$[E_a, F_b] = \delta_{ab}H_b$$
$$[E_a, \ldots [E_a, E_b]] = 0 \qquad\qquad [F_a, \ldots [F_a, F_b]] = 0$$

In the final line there are $(1 - A_{ab})$ E_a's in the first equation and the same number of F_a's in the second equation.

The Kac-Moody algebra may also be formulated in the Cartan-Weyl basis from H_i, E_a and F_a, where H_i is related to the Cartan sub-algebra by

$$H_a = 2\frac{\alpha_a^i H_i}{(\alpha_a, \alpha_a)}$$
$$\Rightarrow \alpha_a \cdot H = \frac{(\alpha_a, \alpha_a)}{2}H_a \tag{6}$$

1.2 Constructing the Algebra

Any $\mathcal{G}^{+++}$ belongs to a class of Lorentzian Kac-Moody algebras considered in [6]. These algebras have the property that their associated Dynkin diagram

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[1]The case of C_{n-3}^{+++} is very cumbersome: to consider the embedding of an A_3 gravity line we must delete no less than $(n-3)$ nodes, which gives rise to $(n-4)^2$ scalar fields and $2n-8$ one-forms [21], while we offer no proof of the element (1) for C_{n-3}^{+++}, we have no reason to believe it will fail in this theory

[2]For a review of Cartan's classification of the finite semisimple Lie groups $\mathcal{G}$ and the corresponding Cartan matrices, A_{ab}, see [15, 16].

possess at least one node which upon the deletion leads to component diagrams which are of finite type except for at most one of affine type. The authors of [6] have studied the properties of the Lorentzian algebras in terms of the algebra associated with the Dynkin diagram that remains after the node deletion. In this paper we may sometimes wish to delete more than one node from some Dynkin diagrams in such a way that the remaining diagram is an A_n diagram. The remaining A_n sub-algebra gives rise to the gravity sector and is indicated in Appendix B on the Dynkin diagrams for the relevant $\mathcal{G}^{+++}$ by solid nodes, we refer to these roots as the gravity line. We use this remaining sub-algebra to investigate the Kac-Moody algebra, $\mathcal{G}^{+++}$. Our generators therefore appear in an $SL(n)$ sub-algebra and are classified by the level at which they arise in the decomposition from the very extended algebra. In this paper the level is the coefficient, or coefficients, of the simple roots that are eliminated from the very extended algebra to obtain a finite $SL(n)$ sub-algebra. Any root, α, may be expressed as a sum of simple roots $\alpha = \sum_a n_a \alpha_a$ where a runs over the simple roots of the very extended algebra. If the $SL(n)$ sub-algebra is formed by removing simple root c then α is a level n_c root; if roots c and d are removed then α has level (n_c, n_d) and so on.

By definition a Kac-Moody algebra is the multiple commutators of the simple root generators, E_a and separately those of F_a, subject to the Serre relations (5). However as a matter of practise this is very difficult to carry out, in this section we explain how to construct the $\mathcal{G}^{+++}$ algebra using the tables of low-order generators in [9]. If we consider two A_n representations with root coefficients $(a_1 a_2 \ldots a_n)$ and $(b_1 b_2 \ldots b_n)$ then their commutator has a root which is the sum, i.e. it has root coefficients $((a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n))$.

The Borel sub-algebra is formed from the Cartan sub-algebra generators, H_a, where $a = 1 \ldots n$, the positive root generators, $K^a{}_b$, where $a < b$ and $a, b = 1 \ldots n$, and the generators, $R^{a_1 \cdots a_n}{}_{b_1 \ldots b_m}$ which arise from our decomposition of $\mathcal{G}^{+++}$. The generators of the Cartan sub-algebra are constructed from the $K^a{}_a$ generators where $a = 1 \ldots n$ and a dilaton generator, which we label R_0, although some of the theories we are interested in have no so-called "dilaton". The dilaton generator is so called because it leads to a dilaton scalar field in a non-linear realisation. The $K^a{}_b$ and $R^{a_1 \cdots a_n}{}_{b_1 \ldots b_m}$ obey the commutation rules

$$
\begin{aligned}
[K^a{}_b, K^c{}_d] &= \delta^b_c K^a{}_d - \delta^a_d K^b{}_c \\
[K^a{}_b, R^{c_1 \cdots c_n}{}_{d_1 \ldots d_m}] &= \delta^{c_1}_b R^{a c_2 \cdots c_n}{}_{d_1 \ldots d_m} + \ldots + \delta^{c_n}_b R^{c_1 \cdots c_{n-1} a}{}_{d_1 \ldots d_m} \\
&\quad - \delta^a_{d_1} R^{c_1 \cdots c_n}{}_{b d_2 \ldots d_m} - \ldots - \delta^a_{d_m} R^{c_1 \cdots c_n}{}_{d_1 \ldots d_{m-1} b}
\end{aligned}
\tag{7}
$$

Where the second equation includes a contribution from the trace of $K^a{}_b$, which is really a $GL(n)$ generator.

Let us make use of the following notation for a generic commutator

$$
[R^{n_1 n_2 \ldots n_a}_{(s_1)}, R^{n_1 n_2 \ldots n_b}_{(s_2)}] = c^{s_1, s_2}_{a,b} R^{n_1 n_2 \ldots n_{a+b}}_{(s_1 + s_2)}
\tag{8}
$$

The s labels are the levels associated with the elimination of one of the simple roots in theories where the decomposition involves the elimination of more than one simple root. It is used to differentiate different types of generators, and can

4

be simply read off from the root. We define which root is associated with the s labels in each of the $\mathcal{G}^{+++}$ that we consider. In $\mathcal{G}^{+++}$ theories where only one root is eliminated the s label is redundant and will be dropped.

Some general restrictions on $c_{a,b}^{s_1,s_2}$ are determined from the general Jacobi identity, where we simplify our notation so that the figure in square brackets is the number of antisymmetrized indices on each operator

$$[R_{s_1}^{[a]}, [R_{s_2}^{[b]}, R_{s_3}^{[c]}]] + [R_{s_2}^{[b]}, [R_{s_3}^{[c]}, R_{s_1}^{[a]}]] + [R_{s_3}^{[c]}, [R_{s_1}^{[a]}, R_{s_2}^{[b]}]] = 0$$
$$\Rightarrow c_{a,b+c}^{s_1,s_2+s_3} c_{b,c}^{s_2,s_3} + c_{b,c+a}^{s_2,s_3+s_1} c_{c,a}^{s_3,s_1} + c_{c,a+b}^{s_3,s_1+s_2} c_{a,b}^{s_1,s_2} = 0 \qquad (9)$$

Having commenced with the generators associated with the simple roots, we construct new generators by taking their commutators as we proceed. Subsequently by taking a commutator with a third generator we find restrictions on our commutator coefficients, $c_{a,b}^{s_1,s_2}$, using the Jacobi identity given above.

We will be particularly interested in finding how the dilaton commutes with other generators, which is denoted by $R_{s_1}^{[a]} = R_0$ to be the dilaton generator. The dilaton generator appears as a member of the Cartan sub-algebra, so it doesn't change the A_n representation of the operator it commutes with. Of particular use are the following specific cases of (9) which give the commutator coefficients for the general commutator $[R_0, R_s^{[m]}] = c_{0,m}^{0,s} R_s^{[m]}$ in a recursive form which we have found by setting $b = 1$ and $c = m - 1$ for the following values of s_2 and s_3 in (9)

$$s_2 = 0, s_3 = 0 \qquad \Rightarrow \qquad c_{0,m}^{0,0} = c_{0,m-1}^{0,0} + c_{0,1}^{0,0} \qquad (10)$$

$$s_2 = 0, s_3 = 1 \qquad \Rightarrow \qquad c_{0,m}^{0,1} = c_{0,m-1}^{0,1} + c_{0,1}^{0,0} \qquad (11)$$

$$s_2 = 1, s_3 = 0 \qquad \Rightarrow \qquad c_{0,m}^{0,1} = c_{0,m-1}^{0,0} + c_{0,1}^{0,1} \qquad (12)$$

$$s_2 = 1, s_3 = 1 \qquad \Rightarrow \qquad c_{0,m}^{0,2} = c_{0,m-1}^{0,1} + c_{0,1}^{0,1} \qquad (13)$$

These relations allow us to find all the commutators with the dilaton, R_0, up to low orders, and we only need to specify one unknown which we choose to be $(c_{0,1}^{0,0})$.[3] Similar recursive commutator relations can be found for higher order generators by considering s labels which are greater than one in equation (9).

Let us consider the example of F_4^{+++} to demonstrate how we build the algebra. The low order generators of F_4^{+++} are given in reference [9] and we

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[3] The relation between $c_{0,1}^{0,1}$ and $c_{0,1}^{0,0}$ can be found by considering the commutation any of the Cartan elements, e.g. H_n, that contain R_0 with the generator for the simple root corresponding the second eliminated root, e.g. for the F_4^{+++} algebra this is E_7, which can be seen in Appendix B on the Dynkin diagram.

list them here with their associated root coefficients in brackets after them.

$$
\begin{array}{lll}
R_0(0000000) & R_1(0000001) \\
R_0^6(0000010) & R_1^6(0000011) \\
R_0^{56}(0000120) & R_1^{56}(0000121) & R_2^{56}(0000122) \hspace{1cm} (14) \\
 & R_1^{456}(0001231) & R_2^{456}(0001232) \\
 & R_1^{3456}(0012341) & R_2^{456,6}(0001242) \\
 & & R_2^{3456}(0012342)
\end{array}
$$

We make use of this bracket notation throughout this paper, it is simply a list of the coefficients n_a when a root, α, associated with a particular generator, is written as a positive sum of simple roots $\alpha = \sum_a n_a \alpha_a$, for example (0000121) corresponds to the root $\alpha = \alpha_5 + 2\alpha_6 + \alpha_7$.

The seven generators associated with the simple roots are $E_a = K^a{}_{a+1}$ for $a = 1 \ldots 5$, $E_6 = R_0^6$ and $E_7 = R_1$. The subscript s labels are the levels corresponding to the simple root R_1. We read off the following commutation relations for the dilaton R_0 with the other generators from the root coefficients

$$
\begin{array}{lll}
[R_0, R_0^a] = c_{0,1}^{0,0} R_0^a & [R_0, R_1^a] = c_{0,1}^{0,1} R_1^a \\
[R_0, R_0^{ab}] = c_{0,2}^{0,0} R_0^{ab} & [R_0, R_1^{ab}] = c_{0,2}^{0,1} R_1^{ab} & [R_0, R_2^{ab}] = c_{0,2}^{0,2} R_2^{ab} \\
 & [R_0, R_1^{abc}] = c_{0,3}^{0,1} R_1^{abc} & [R_0, R_2^{abc}] = c_{0,3}^{0,2} R_2^{abc} \hspace{0.5cm} (15) \\
 & [R_0, R_1^{abcd}] = c_{0,4}^{0,1} R_1^{abcd} & [R_0, R_2^{abc,d}] = c_{0,4}^{0,2} R_2^{abc,d}
\end{array}
$$

By considering $[H_7, E_7] = 2E_7$ and $[H_7, E_6] = -E_6$, from equation (5), we find that $c_{0,1}^{0,1} = -c_{0,1}^{0,0}$ and using equations (10)-(13) we find that in terms of $c_{0,1}^{0,0}$ the commutation relations above are

$$
\begin{array}{lll}
[R_0, R_0^a] = c_{0,1}^{0,0} R_0^a & [R_0, R_1^a] = -c_{0,1}^{0,0} R_1^a \\
[R_0, R_0^{ab}] = 2c_{0,1}^{0,0} R_0^{ab} & [R_0, R_1^{ab}] = 0 & [R_0, R_2^{ab}] = -2c_{0,1}^{0,0} R_2^{ab} \\
 & [R_0, R_1^{abc}] = c_{0,1}^{0,0} R_1^{abc} & [R_0, R_2^{abc}] = -c_{0,1}^{0,0} R_2^{abc} \\
 & & \hspace{2cm} (16) \\
 & [R_0, R_1^{abcd}] = 2c_{0,1}^{0,0} R_1^{abcd} & [R_0, R_2^{abc,d}] = 0
\end{array}
$$

We fix $c_{0,0}^{0,1}$ to be $\frac{1}{\sqrt{8}}$ and obtain the relations given later in equation (90). Other commutators can easily be found using the table of roots given in [9] together with the Jacobi identity (9) and the Serre relations (5).

1.3 Highest and Lowest Weights in $\mathcal{G}^{+++}$ Theories

In any finite dimensional semi-simple Lie algebra $\mathcal{G}$ the highest weight Λ_H and lowest weight Λ_L of any representation are related by the formula $\Lambda_L = S_0 \Lambda_H$ where S_0 is the longest element of the Weyl group of $\mathcal{G}$. We recall that any Weyl group element S can be expressed as a product of Weyl group elements $S_a = S_{\alpha_a}$ corresponding to the reflections in the simple roots α_a of $\mathcal{G}$. There are a number of ways to write any S involving different numbers of S_a, but there is

obviously always a way that involves the smallest number of S_a's. This smallest number of S_a's required is called the length of the Weyl group element S. It turns out that the Weyl group element with the maximum length is unique and this is the one that relates the highest and lowest weights. The Weyl group element S_b acts on the simple roots by

$$S_b \alpha_a = (s)_a{}^b \alpha_b = \alpha_a - 2\frac{(\alpha_b, \alpha_a)}{(\alpha_b, \alpha_b)}\alpha_b \tag{17}$$

and on the elements of the Cartan sub-algebra H_a by $S_b H_a = (s)_a{}^b H_b$. Hence the transformation of the the Cartan sub-algbra elements by the longest element of the Weyl group is given by $S_0 H_a = (s_0)_a{}^b H_b$

For A_n the action of the Weyl group on the Cartan sub-algebra is given by

$$S_a H_b = H_b \ b \neq a, \ b \neq a \pm 1, \ S_n H_n = -H_n, \ S_a H_{a\pm 1} = H_a + H_{a+1} \tag{18}$$

In terms of the generators $K^a{}_b$ of A_n the Cartan sub-algebra elements are given by $H_a = K^a{}_a - K^{a+1}{}_{a+1}$ and the effect of the Weyl group element S_a is

$$K^a{}_a \ \leftrightarrow \ K^{a+1}{}_{a+1} \tag{19}$$

all other elements being left inert. The longest Weyl group element of A_n is given by $S_0 = (S_1 \ldots S_n)(S_1 \ldots S_{n-1})\ldots(S_1 S_2)(S_1)$.

In this paper we will consider the generators of G^{+++} decomposed level by level in terms of representations of A_n for suitable n. The highest A_n weight generators in $\mathcal{G}^{+++}$ and their $\mathcal{G}^{+++}$ roots are listed in reference [9]. The highest A_n weight generators have $SL(n+1)$ indices that that are the highest possible. For example, the generator R^a belong to the one rank tensor representation of $SL(n+1)$ and the generator corresponding to the highest A_n weight is R^{n+1}. The generator corresponding to lowest A_n weight is proportional to R^1 and one can verify that the highest and lowest A_n weights are indeed related by the action of S_0.

As we are dealing with a non-linear realisation each generator in Borel sub-algebra is associated with a field. However, we will be interested in electric branes which couple to fields with a 1 index and so correspond to lowest A_n weight generators in G^{+++}. In equation 1 we will also require the $\mathcal{G}^{+++}$ root and associated Cartan sub-algebra element that corresponds to the lowest weight of A_n. We can find this using the action of S_0 as described above. The reader may verify that this has the same effect as applying $K^a{}_b$ generators to the generators of G^{+++} to go between the highest and lowest A_n weight generators.

1.4 Solutions in Generic Gravity Theories

There is a very substantial literature on brane solutions in generic supergravity theories [13], for a review see [17, 18]. We use the notation of [17] to express the general equations of motion, line element and dilaton that arise from a theory containing gravity, a gauge field and a dilaton, which is the truncation of the

full supergravity action, and allows us to investigate single brane solutions. The generic action integral is

$$A = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} (R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.n_i!}e^{a_i\phi}F_{a_1...a_{n_i}}F^{a_1...a_{n_i}}) \quad (20)$$

Where D is the dimension of the background spacetime, a_i is the dilaton coupling constant, $F_{a_1...a_{n_i}}$ is a general n_i-form field strength. We note here that the equations of motion derived from a truncated action, as above, is always consistent with the single electric brane solutions presented here. Chern-Simons terms in the full action will alter the equation of motion obtained from varying the gauge potentials. For our solutions the only non-zero gauge potentials possess a timelike index, any remaining Chern-Simons-like terms in the gauge equation will be a wedge product of such gauge fields and so are identically zero. Consequently the Chern-Simons terms do not effect our single electric brane solutions as they vanish at the level of the equations of motion. The equations of motion coming from the general action above are

$$R^\mu{}_\nu = \frac{1}{2}\partial^\mu\phi\partial_\nu\phi + \frac{1}{2n_i!}e^{a_i\phi}(n_i F^{\mu a_2...a_{n_i}}F_{\nu a_2...a_{n_i}} - \frac{n_i-1}{D-2}\delta^\mu{}_\nu F_{a_1...a_{n_i}}F^{a_1...a_{n_i}})$$

$$\partial_\mu(\sqrt{-g}\partial^\mu\phi) = \frac{\sqrt{-g}.a_i}{2.n_i!}e^{a_i\phi}F_{a_1...a_{n_i}}F^{a_1...a_{n_i}} \quad (21)$$

$$\partial_\mu(\sqrt{-g}e^{a_i\phi}F^{\mu a_2...a_{n_i}}) = 0$$

A theory containing such field strengths, $F_{a_1...a_{n_i}}$, has conserved charges which are associated with $(n_i - 2)$-branes, and their dual field strengths are associated to $(D - n_i - 2)$-branes. A BPS p-brane solution with arbitrary dilaton coupling has line element

$$ds^2 = N_p^{-2(\frac{D-p-3}{\Delta})}(-dy_1^2 + dx_2^2 + \ldots dx_{p+1}^2) + N_p^{2(\frac{p+1}{\Delta})}(dx_{p+2}^2 + \ldots dx_D^2) \quad (22)$$

Where $\Delta = (p+1)(D-p-3) + \frac{1}{2}a_i^2(D-2)$, a_i is the dilaton coupling constant of the associated field strength, $F_{a_1...a_{n_i}}$, and N_p is an harmonic function, equal to (2), and taking the general form

$$N_p = 1 + \frac{1}{D-p-3}\sqrt{\frac{\Delta}{2(D-2)}}\frac{\|\mathbf{Q}\|}{r^{(D-p-3)}} \quad (23)$$

Where $\mathbf{Q}$ is the conserved charge associated with the p-brane solution, and $r^2 = x_{p+2}^2 + \ldots x_D^2$. We are able to read from any given line element the value of the dilaton coupling constant, a_i, to within a minus sign. The associated dilaton, for coupling constant a_i, is

$$e^\phi = N_p^{a_i\frac{D-2}{\Delta}} \quad (24)$$

A second brane solution is associated with the dual of the field strength that gives rise to the p-brane solution. This is a $(D-p-4)$-brane, where the dilaton coupling constant, $-a_i$, is now the negative of that for the p-brane. We find that for the $(D-p-4)$-brane

$$\Delta' = (D-p-3)(p+1) + \frac{1}{2}a_i^2(D-2) = \Delta \quad (25)$$

8

And the line element for the brane associated with the dual field strength is

$$ds^2 = N_{D-p-4}^{-2(\frac{p+1}{\Delta})}(-dy_1^2 + dx_2^2 + \ldots dx_{D-p-3}^2) + N_{D-p-4}^{2(\frac{D-p-3}{\Delta})}(dx_{D-p-2}^2 + \ldots dx_D^2)$$ (26)

The associated dilaton is

$$e^\phi = N_{D-p-4}^{-a_i \frac{D-2}{\Delta}}$$ (27)

Ensuring that the line elements are related as (22) is to (26) and confirmation of a change of sign in the power of the harmonic function N in the dilaton, as between (24) and (27), enables us to recognise which field strengths are dual to each other in the $\mathcal{G}^{+++}$ considered in this paper.

1.5 The Dilaton Generator

Equation (1) plays a central role in our work and may contain the dilaton generator, R_0, through $\beta \cdot H$ for particular algebras. The non-linear realisation allows us to write down the most generally covariant field equations for the field content arising from a particular $\mathcal{G}^{+++}$ in which the dilaton field, A appears within a factor in the field strength. The procedure for finding the field equations is given in [2, 1, 19] and, using the above commutator relations, in the non-linear realisation of $\mathcal{G}^{+++}$ the field strengths take the form

$$F_{a_1 a_2 \ldots a_p \, s_1} = p e^{-c_{0,p-1}^{0,s_1} A}(\partial_{[a_1} A_{a_2 \ldots a_p]_{s_1}} + \ldots)$$ (28)

Where "$+ \ldots$" indicates terms which are not total derivatives but have the correct number of indices and within which the sum of s labels matches the s label of the field strength.

The required dual field strength in the non-linear realisation is found to take the form

$$F_{a_1 a_2 \ldots a_{(D-p)} \, s_2} = (D-p) e^{-c_{0,D-(p-1)}^{0,s_2} A}(\partial_{[a_1} A_{a_2 \ldots a_{D-p}]_{s_2}} + \ldots)$$ (29)

Where D is the number of background spacetime dimensions, and we construct the remaining field strengths similarly. Having constructed the field strengths for a particular theory, we follow the process in [2, 1, 19] and write down the equations relating a field strength and its Hodge dual, which take the general form

$$*F_{a_1 a_2 \ldots a_{(D-p)} \, s_1} \equiv F_{a_1 a_2 \ldots a_p \, s_1} = \frac{1}{p!} \epsilon_{a_1 a_2 \ldots a_D} F_{s_2}^{a_{(p+1)} \ldots a_D}$$ (30)

Where the Hodge dual is indicated by *, and $\epsilon_{a_1 a_2 \ldots a_D}$ is the usual completely antisymmetric tensor.

We then substitute our expressions for $F_{a_1 a_2 \ldots a_p \, s_1}$ and $F_{a_1 a_2 \ldots a_{D-p} \, s_2}$ into this equation. The second order field equations are obtained by bringing all factors containing the dilaton field e^{kA} together as a coefficient of the field strength appearing in the Lagrangian and differentiating the equation. Any total derivatives vanish by the Bianchi identity and we obtain an equation of the form

$$\partial_{[a_m} * (p e^{(-c_{0,p-1}^{0,s_1} + c_{0,D-(p-1)}^{0,s_2})A}(\partial_{a_1} A_{a_2 \ldots a_p]_{s_1}}) + \ldots) = \partial_{[a_m}(+ \ldots)]$$ (31)

9

The "+..." correspond to the non-total derivative terms coming from our non-linear formulation of the field strengths. The coefficient of the total derivative on the left-hand-side is now exactly that which appears in a Lagrangian formulation of this system, where the field strength is just the total derivative, $p\partial_{[a_1}A_{a_2...a_p]}$. These considerations are useful in making comparisons with the literature, in particular with [11]. The reader may see a fully worked example of this method for D_{n-3}^{+++} in section 2.

2 Simply Laced Groups

A simply laced group has simple roots which all have the same length, here our roots are normalised such that $(\alpha_a, \alpha_a) = 2$.

2.1 Very Extended D_{n-3}

D_{24}^{+++} is the symmetry underlying the effective action of the bosonic string theory [1], and the analogous n-dimensional generalisation is D_{n-3}^{+++} [1, 7], these are maximally oxidised theories. The Dynkin diagram for D_{n-3}^{+++} is shown in Appendix B, where the solid nodes indicate the gravity line we shall consider, which is an A_{n-2} sub-algebra.

We decompose the D_{n-3}^{+++} algebra with respect to its A_{n-2} sub-algebra, the simple roots whose nodes we delete are α_{n-1} and α_n, as enumerated in Appendix B and we associate the levels l_1 and l_2 with these respectively. Our s labels are chosen to be the l_2 level that the generator appears at. We find the generators $K^a{}_b$ at level $(0,0)$, corresponding to the A_{n-2} sub-algebra of the gravity line, and the following other generators up to level $(1,1)$

$$
\begin{array}{lllll}
l_1 \longrightarrow & & 0 & 1 & \\
l_2 \downarrow & 0 & R_0 & R_1^{a_1 a_2 ... a_{(n-5)}} & \\
& 1 & R_0^{ab} & R_1^{a_1 a_2 ... a_{(n-4)}, b} & (32) \\
& & & R_1^{a_1 a_2 ... a_{(n-3)}} &
\end{array}
$$

The so-called dual to gravity $R_1^{a_1 a_2 ... a_{(n-4)}, b}$ is listed amongst our low-level generators for completeness but it is not as well understood as the other listed generators and we will not consider it here, nor throughout this paper in any of the $\mathcal{G}^{+++}$ theories that we consider, as a starting point generator for finding the encoded brane solutions via (1). Each of the generators is associated with a root, β, such that $(\beta, \beta) = 2$ except for the dilaton generator R_0 and the generator $R_1^{a_1 a_2 ... a_{(n-3)}}$, for which $(\beta, \beta) = 0$. That $(\beta, \beta) = 0$ for these generators means that we cannot commence with the generators R_0 and $R_1^{a_1 a_2 ... a_{(n-3)}}$ and use the group element in equation (1) to deduce an electric brane associated with them, as we would have a singularity coming from the $\frac{1}{(\beta, \beta)}$ for these generators. As such they are discarded as starting points for our method, as are all such generators which possess such a singularity in equation (1).

We make use of the following commutators, where we have chosen the coefficient $\frac{24}{D-2}$ in keeping with [1] and the other commutator has been determined from the Serre relations (5)

$$[R_0, R_0^{ab}] = \frac{24}{D-2} R_0^{ab}$$

$$[R_0, R_1^{a_1 a_2 \ldots a_{n-5}}] = -\frac{24}{D-2} R_1^{a_1 a_2 \ldots a_{n-5}} \tag{33}$$

The simple root generators of D_{n-3}^{+++} are

$$E_a = K^a{}_{a+1}, a = 1, \ldots (n-2), E_{(n-1)} = R_0^{(n-2)(n-1)}, E_n = R_1^{56 \ldots (n-1)} \tag{34}$$

and the Cartan sub-algebra generators, H_a, are given by

$$H_a = K^a{}_a - K^{a+1}{}_{a+1}, a = 1, \ldots (n-2),$$
$$H_{(n-1)} = -\tfrac{2}{D-2}(K^1{}_1 + \ldots K^{n-3}{}_{n-3}) + \tfrac{D-4}{D-2}(K^{n-2}{}_{n-2} + K^{n-1}{}_{n-1}) + \tfrac{1}{6}R_0$$
$$H_n = -\tfrac{D-4}{D-2}(K^1{}_1 + \ldots K^4{}_4) + \tfrac{2}{D-2}(K^5{}_5 + \ldots K^{n-1}{}_{n-1}) - \tfrac{1}{6}R_0 \tag{35}$$

The low-level field content [9] is $\hat{h}_a{}^b$, A, $A_{a_1 a_2}$ and their field strengths have duals derived from $A_{a_1 \ldots a_{n-4}, b}$, $A_{a_1 \ldots a_{n-3}}$, $A_{a_1 \ldots a_{n-5}}$ respectively. Our choice of local sub-algebra for the non-linear realisation allows us to write the group element as

$$g = \exp(\sum_{a \leq b} \hat{h}_a{}^b K^a{}_b) \exp(\frac{1}{(n-3)!} A_{a_1 a_2 \ldots a_{n-3}} R_1^{a_1 a_2 \ldots a_{n-3}})$$

$$\exp(\frac{1}{(n-4)!} A_{a_1 a_2 \ldots a_{n-4}, b} R_1^{a_1 a_2 \ldots a_{n-4}, b}) \tag{36}$$

$$\exp(\frac{1}{(n-5)!} A_{a_1 a_2 \ldots a_{n-5}} R_1^{a_1 a_2 \ldots a_{n-5}})$$

$$\exp(\frac{1}{2!} A_{ab} R_0^{ab}) \exp(A R_0)$$

The field content of the associated maximally oxidised theory given in [11] agrees with the low-order D_{n-3}^{+++} content given above. The Lagrangian for the oxidised theory, contains a graviton, ϕ, a dilaton, A, and a 3-form field strength, $F_{\mu\nu\rho} = 3\partial_{[\mu} A_{\nu\rho]}$, and is given by

$$A = \frac{1}{16\pi G_{n-1}} \int d^{n-1}x \sqrt{-g}(R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2.3!} e^{\sqrt{\frac{8}{D-2}}\phi} F_{\mu\nu\rho} F^{\mu\nu\rho}) \tag{37}$$

We now demonstrate the use of the group element given in equation (1) in finding the electric branes of D_{n-3}^{+++} from the generators listed in equation (32). We find the following electric branes

A String We commence by setting β in (1) to be the root associated with the generator, $R_0^{(n-2)(n-1)}$ $(00 \ldots 010)$, the highest weight in the $R^{a_1 a_2}$ representation, corresponding to the field A_{a1a2} in [9]. Our motivation is to find the usual BPS electric branes that emerge using the group element given. As such we wish to work with representations that have a time-like index, which we may

11

obtain from our highest weight via multiple commutation with the appropriate $K^a{}_b$ generators such that all the indices on our operator are lowered to the lowest consecutive sequence of indices, including the time-like index. This has the consequence of finding the lowest weight generator in a particular representation. Our corresponding lowest weight in this representation is $R_0^{12}(12\ldots2110)$. The Cartan sub-algebra element associated with this lowest weight is found by using the root coefficients of the lowest weight, $(12\ldots2110)$, as coefficients of the Cartan sub-algebra elements associated with each root in the expansion of $\beta \cdot H$ so that

$$
\begin{aligned}
\beta \cdot H =& H_1 + 2(H_2 + \ldots H_{n-3}) + H_{n-2} + H_{n-1} \\
=& (K^1{}_1 - K^2{}_2) + 2(K^2{}_2 - K^{n-2}{}_{n-2}) + (K^{n-2}{}_{n-2} - K^{n-1}{}_{n-1}) \\
& - \frac{2}{D-2}(K^1{}_1 + \ldots K^{n-3}{}_{n-3}) + \frac{D-4}{D-2}(K^{n-2}{}_{n-2} + K^{n-1}{}_{n-1}) \\
& + \frac{1}{6}R_0 \\
=& \frac{D-4}{D-2}(K^1{}_1 + K^2{}_2) - \frac{2}{D-2}(K^3{}_3 + \ldots K^{n-1}{}_{n-1}) + \frac{1}{6}R_0 \qquad (38)
\end{aligned}
$$

We now substitute equation (38) into our group element given in equation (1), noting that as D_{n-3}^{+++} is a simply laced group $(\beta,\beta) = 2$. The corresponding group element is

$$
\begin{aligned}
g_A =& \exp(-\frac{1}{2}\ln N_1(\frac{D-4}{D-2}(K^1{}_1 + K^2{}_2) - \frac{2}{D-2}(K^3{}_3 + \ldots K^{n-1}{}_{n-1}) \\
& + \frac{1}{6}R_0))\exp((1-N_1)E_\beta) \\
=& \exp(-\frac{1}{2}\ln N_1(\frac{D-4}{D-2}(K^1{}_1 + K^2{}_2) - \frac{2}{D-2}(K^3{}_3 + \ldots K^{n-1}{}_{n-1}))) \\
& \exp(N_1^{-\frac{2}{D-2}}(1-N_1)E_\beta)\exp(-\frac{1}{12}\ln N_1 R_0) \qquad (39)
\end{aligned}
$$

Where we have moved the generator associated with the dilaton, R_0, to the right so that it agrees with the structure of the group element from which the non-linear realisation is constructed in equation (36). We make use of $[R_0, R_0^{ab}]$ in equation (33) to do this and by examining the resulting R_0 term we find a dilaton which is given by

$$
e^A = N_1^{-\frac{1}{12}} \qquad (40)
$$

By reading off the coefficients of the $K^a{}_a$ in the group element we find a line element corresponding to a string

$$
ds^2 = N_1^{-\frac{D-4}{D-2}}(-dy_1^2 + dx_2^2) + N_1^{\frac{2}{D-2}}(dx_3^2 + \ldots + dx_{n-1}^2) \qquad (41)
$$

The brane is derived from a gauge potential which we can also read off from equation (39) as the coefficient of E_β

$$
A^T_{12(0)} = N_1^{-\frac{2}{D-2}}(1-N_1) \qquad (42)
$$

This has tangent space indices, which we indicate with a T. To make the change to world volume indices we use the vierbein which we deduce from equation (41),

$(e^{\hat{h}})^1{}_1 = (e^{\hat{h}})^2{}_2 = N_1^{-\frac{D-4}{2(D-2)}}$ and find

$$A_{12(0)} = N_1^{-1} - 1 \tag{43}$$

This gives rise to a 3-form field strength, which we label $F_{\mu\nu\rho 0}$.

An $(n-6)$ Brane Let us consider the representation $R^{a_1 a_2 \ldots a_{n-5}}$, corresponding to the field $A_{a_1 a_2 \ldots a_{n-5}}$ in [9] whose highest weight generator is $R_1^{5 \ldots (n-1)}$ $(00 \ldots 01)$. We take β in equation (1) to be the root associated with the highest weight generator. The lowest weight generator in the representation is $R_1^{1 \ldots (n-5)}$ $(1234 \ldots 432101)$ and the corresponding element in the Cartan sub-algebra is

$$\begin{aligned}
\beta \cdot H =& H_1 + 2H_2 + 3H_3 + 4(H_4 + \ldots H_{n-5}) + 3H_{n-4} + 2H_{n-3} + H_{n-2} + H_n \\
=& K^1{}_1 + K^2{}_2 + K^3{}_3 + K^4{}_4 - K^{n-4}{}_{n-4} - K^{n-3}{}_{n-3} - K^{n-2}{}_{n-2} \\
& - K^{n-1}{}_{n-1} - \frac{D-4}{D-2}(K^1{}_1 + \ldots K^4{}_4) + \frac{2}{D-2}(K^5{}_5 + \ldots K^{n-1}{}_{n-1}) \\
& - \frac{1}{6}R_0 \\
=& \frac{2}{D-2}(K^1{}_1 + \ldots K^{n-5}{}_{n-5}) - \frac{D-4}{D-2}(K^{n-4}{}_{n-4} + \ldots K^{n-1}{}_{n-1}) \quad (44) \\
& - \frac{1}{6}R_0
\end{aligned}$$

Substituting this into equation (1), recalling that $(\beta,\beta) = 2$, we find the corresponding group element is

$$\begin{aligned}
g_A =& \exp(-\frac{1}{2}\ln N_{n-6}(\frac{2}{D-2}(K^1{}_1 + \ldots K^{n-5}{}_{n-5}) \\
& - \frac{D-4}{D-2}(K^{n-4}{}_{n-4} + \ldots K^{n-1}{}_{n-1}) - \frac{1}{6}R_0)) \exp((1 - N_{n-6})E_\beta) \\
g_A =& \exp(-\frac{1}{2}\ln N_{n-6}(\frac{2}{D-2}(K^1{}_1 + \ldots K^{n-5}{}_{n-5}) \quad (45) \\
& - \frac{D-4}{D-2}(K^{n-4}{}_{n-4} + \ldots K^{n-1}{}_{n-1}))) \exp(N_{n-6}^{-\frac{2}{D-2}}(1 - N_{n-6})E_\beta) \\
& \exp(\frac{1}{12}\ln N_{n-6}R_0)
\end{aligned}$$

Where in the last line we have made use of $[R_0, R_1^{a_1 \ldots a_{n-5}}]$, given in equation (33) to move the dilaton generator, R_0, to the far right of the expression in agreement with the group element from which the non-linear realisation is constructed (36). By examining the R_0 term we find a dilaton given by

$$e^A = N_{n-6}^{\frac{1}{12}} \tag{46}$$

And a line element corresponding to an electric $(n-6)$-brane

$$ds^2 = N_{n-6}^{-\frac{2}{D-2}}(-dy_1^2 + \ldots dx_{n-5}^2) + N_{n-6}^{\frac{D-4}{D-2}}(dx_{n-4}^2 + \ldots + dx_{n-1}^2) \tag{47}$$

The brane is derived from a gauge potential

$$A^T_{12 \ldots (n-5)(1)} = N_{n-6}^{-\frac{2}{D-2}}(1 - N_{n-6}) \tag{48}$$

We use $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^{n-5}{}_{n-5} = N_{n-6}^{-\frac{1}{D-2}}$ and make the change to world volume indices

$$
\begin{aligned}
A_{12\ldots(n-5)(1)} &= N_{n-6}^{-\frac{2}{D-2}} N_{n-6}^{-\frac{n-5}{D-2}}(1 - N_{n-6}) \\
&= N_{n-6}^{-1} - 1
\end{aligned}
\tag{49}
$$

Where we have used $D = n - 1$ to get to the second line. This gauge potential is associated with an $(n-4)$-form field strength which we conclude is the dual of $F_{\mu\nu\rho 0}$.

We have have successfully reproduced all the usual BPS electric branes using the group element given in equation (1). The formulae we have found above do indeed correspond to the solutions of the Lagrangian (37).

Let us now consider a fully worked example of the method outlined in section 1 to find a relation between our dilaton generator A and ϕ. We have found the gauge potentials $A^{(0)}, A^{(0)}_{a_1 a_2}$ and $A^{(1)}_{a_1 \ldots a_{n-5}}$ and we have one 3-form field strength appearing in the Lagrangian of equation (37). Using equation (28) we write down the most general equation for the 3-form field strength

$$
F_{a_1 a_2 a_3 (0)} = 3e^{-c_{0,2}^{0,0} A} \partial_{[a_1} A_{a_2 a_3] (0)}
\tag{50}
$$

Its dual is an $(n-4)$-form field strength which is most generally

$$
F_{a_1 \ldots a_{n-4} (1)} = (n-4)e^{-c_{0,n-5}^{0,1} A} \partial_{[a_1} A_{a_2 \ldots a_{n-5}] (1)}
\tag{51}
$$

These are related by duality giving

$$
*F_{a_1 a_2 a_3 (0)} = \frac{1}{(n-4)!} \epsilon_{a_1 a_2 a_3 b_1 \ldots b_{n-4}} F^{a_1 \ldots a_3}{}_{(0)} = F_{b_1 \ldots b_{n-4} (1)}
\tag{52}
$$

Substituting equations (50) and (51) into equation (52) yields

$$
*(3e^{-c_{0,2}^{0,0} A} \partial_{[a_1} A_{a_2 a_3] (0)}) = (n-4)e^{-c_{0,n-5}^{0,1} A} \partial_{[a_1} A_{a_2 \ldots a_{n-4}] (1)}
\tag{53}
$$

If we bring the exponentials of A together on the left-hand-side with the 3-form and differentiate we obtain

$$
\partial_{[a_m} * (3e^{(-c_{0,2}^{0,0} + c_{0,n-5}^{0,1}) A} \partial_{a_1} A_{a_2 a_3] (0)}) = 0
\tag{54}
$$

Where the right-hand-side has vanished under differentiation by the Bianchi identity. The commutation of H_{n-1} given in equation (35) with the positive root generators E_n and E_{n-1} enables us to find a relation between $c_{0,n-5}^{0,1}$ and $c_{0,2}^{0,0}$, we take $l = \frac{1}{12}$ and we find

$$
\begin{aligned}
[H_{n-1}, E_n] &= [H_{n-1}, R_1^{56\ldots(n-1)}] = 0 \\
&\Rightarrow l c_{0,n-5}^{0,1} = -\frac{4}{D-2} \\
[H_{n-1}, E_{n-1}] &= [H_{n-1}, R_0^{(n-2)(n-1)}] = 2R_0^{(n-2)(n-1)} \\
&\Rightarrow l c_{0,2}^{0,0} = \frac{4}{D-2} \\
\Rightarrow \quad c_{0,2}^{0,0} &= -c_{0,n-5}^{0,1}
\end{aligned}
\tag{55}
$$

14

Substituting this identity into our field equation (54) gives

$$\partial_{[a_m} * (3e^{(-2c_{0,2}^{0,0})A} \partial_{a_1} A_{a_2 a_3]_{(0)}}) = 0 \tag{56}$$

We have fixed our structure constants such that $c_{0,2}^{0,0} = \frac{24}{D-2}$ in accordance with reference [1] and by comparison with (37) we find that

$$A = -\frac{1}{12}\sqrt{\frac{D-2}{2}}\phi \tag{57}$$

We apply the method to E_6^{+++} and E_7^{+++} in Appendix A and show similarly that the low order field content derived from the group element (1) matches the field content given in reference [11] for the corresponding oxidised theories.

3 Non-Simply Laced Groups

The general method is slightly less straightforward for non-simply laced groups in that it becomes possible for the roots we find from the group theory to have a norm squared that differs from two. This difference manifests itself in two places. The first arises when we find the element in the Cartan sub-algebra corresponding to the lowest weight in a representation, at this point we must be wary of equation (6) which indicates that we don't have an identity between the root coefficients and the Cartan sub-algebra coefficients but a proportionality factor of $\frac{(\alpha,\alpha)}{2}$. So that, for example, an element of the Cartan sub-algebra corresponding to a root with norm squared half that of the gravity line has an additional factor of $\frac{1}{2}$ in front of it. The second place we must be wary of the varying size of the simple roots comes when we make use of the group element where we must remember to put in the correct value of (β, β) for the generator we are considering.

3.1 Very Extended B_{n-3}

The Dynkin diagram for B_{n-3}^{+++} is shown in Appendix B, where the solid nodes indicate the gravity line we shall consider, which in this case is an A_{n-2} sub-algebra.

We decompose the B_{n-3}^{+++} algebra with respect to its A_{n-2} sub-algebra by deleting the nodes corresponding to the simple roots α_n and α_{n-1} which have generators $R_1^{56\ldots(n-1)}(00\ldots01)$ and $R_0^{(n-1)}(00\ldots10)$ respectively. We associate the levels l_1 and l_2 with these respectively and the s labels on our generators are chosen to be the l_1 level of that generator. From reference [9] we find the B_{n-3}^{+++} algebra contains the generators $K^a{}_b$ at level $(0,0)$, corresponding to the A_{n-2} sub-algebra of the gravity line. We have the following other generators

15

up to level $(1, 2)$ [9]

$$
\begin{array}{cccc}
l_1 \longrightarrow & & 0 & 1 \\
l_2 \downarrow & 0 & R_0 & R_1^{a_1 a_2 \ldots a_{(n-5)}} \\
& 1 & R_0^a & R_1^{a_1 a_2 \ldots a_{(n-4)}} \\
& 2 & R_0^{a_1 a_2} & R_1^{a_1 a_2 \ldots a_{(n-4)}, b} \\
& & & R_1^{a_1 a_2 \ldots a_{(n-3)}}
\end{array}
\tag{58}
$$

The generators R_0^a and $R_1^{a_1 a_2 \ldots a_{n-4}}$ have associated roots β such that $(\beta, \beta) = 1$, the rest and the gravity line nodes have $(\beta, \beta) = 2$, with the exception of the dilaton generator R_0 and the generator $R_1^{a_1 a_2 \ldots a_{(n-3)}}$ which have $(\beta, \beta) = 0$. We cannot apply our method to R_0 and $R_1^{a_1 a_2 \ldots a_{(n-3)}}$ so they are discarded as starting points when we come to find the electric branes from the generators.

We make use of the following commutator relations where we have chosen the commutator coefficient of $[R_0, R_0^{a_1}]$ and the rest have followed from the relations (10)-(13) and the Serre relations (5)

$$
\begin{aligned}
[R_0, R_0^{a_1}] &= \frac{1}{4} \sqrt{\frac{8}{D-2}} R_0^{a_1} \\
[R_0, R_0^{a_1 a_2}] &= \frac{1}{2} \sqrt{\frac{8}{D-2}} R_0^{a_1 a_2} \\
[R_0, R_1^{a_1 a_2 \ldots a_{n-5}}] &= -\frac{1}{2} \sqrt{\frac{8}{D-2}} R_1^{a_1 a_2 \ldots a_{n-5}} \\
[R_0, R_1^{a_1 a_2 \ldots a_{n-4}}] &= -\frac{1}{4} \sqrt{\frac{8}{D-2}} R_1^{a_1 a_2 \ldots a_{n-4}}
\end{aligned}
\tag{59}
$$

We note from content of the table given in [9] for B_{n-3}^{+++} that no higher order commutators are needed.

The simple root generators of B_{n-3}^{+++} are

$$
E_a = K^a{}_{a+1}, a = 1, \ldots (n-2), E_{(n-1)} = R_0^{(n-1)}, E_n = R_1^{56 \ldots (n-1)}
\tag{60}
$$

and the Cartan sub-algebra generators, H_a are given by

$$
\begin{aligned}
H_a &= K^a{}_a - K^{a+1}{}_{a+1}, a = 1, \ldots (n-2), \\
H_{(n-1)} &= -\frac{2}{D-2}(K^1{}_1 + \ldots K^{n-2}{}_{n-2}) + 2\frac{D-3}{D-2}(K^{n-1}{}_{n-1}) \\
&\quad + \sqrt{\frac{8}{D-2}} R_0, \\
H_n &= -\frac{D-4}{D-2}(K^1{}_1 + \ldots K^4{}_4) + \frac{2}{D-2}(K^5{}_5 + \ldots K^{n-1}{}_{n-1}) \\
&\quad - \sqrt{\frac{8}{D-2}} R_0
\end{aligned}
\tag{61}
$$

The low-level field content [9] is $\hat{h}_a{}^b$, A, A_{a_1}, $A_{a_1 a_2}$ and their field strengths have duals derived from $A_{a_1 \ldots a_{n-4}, b}$, $A_{a_1 \ldots a_{n-3}}$, $A_{a_1 \ldots a_{n-4}}$, $A_{a_1 \ldots a_{n-5}}$ respectively. Our choice of local sub-algebra for the non-linear realisation allows us to

16

write the group element of B^{+++}_{n-3} as

$$g = \exp(\sum_{a \le b} \hat{h}_a{}^b K^a{}_b) \exp(\frac{1}{(n-3)!} A_{a_1 a_2 \ldots a_{n-3}} R_1^{a_1 a_2 \ldots a_{n-3}})$$

$$\exp(\frac{1}{(n-4)!} A_{a_1 a_2 \ldots a_{n-4},b} R_1^{a_1 a_2 \ldots a_{n-4},b})$$

$$\exp(\frac{1}{(n-4)!} A_{a_1 a_2 \ldots a_{n-4}} R_1^{a_1 a_2 \ldots a_{n-4}}) \quad (62)$$

$$\exp(\frac{1}{(n-5)!} A_{a_1 a_2 \ldots a_{n-5}} R_1^{a_1 a_2 \ldots a_{n-5}})$$

$$\exp(\frac{1}{2!} A_{ab} R_0^{ab}) \exp(A_a R_0^a) \exp(A R_0)$$

The field content of the associated maximally oxidised theory given in [11] agrees with the low-order B^{+++}_{n-3} content given above. The Lagrangian for the oxidised theory, containing a graviton, ϕ, a 2-form field strength, $F_{\mu\nu\,0} = 2\partial_{[\mu} A_{\nu]0}$, a 3-form field strength, $F_{\mu\nu\rho\,0} = 3(\partial_{[\mu} A_{\nu\rho]0} + A_{[\mu_0} \partial_\nu A_{\rho]0})$ and a dilaton, A, is

$$A = \frac{1}{16\pi G_{n-1}} \int d^{n-1}x \sqrt{-g}(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.2!}e^{\frac{1}{2}\sqrt{\frac{8}{D-2}}\phi} F_{\mu\nu\,0} F_0^{\mu\nu} \quad (63)$$

$$- \frac{1}{2.3!}e^{\sqrt{\frac{8}{D-2}}\phi} F_{\mu\nu\rho\,0} F_0^{\mu\nu\rho})$$

We now demonstrate the use of the group element (1) to generate all of the usual BPS electric branes of B^{+++}_{n-3} starting from the generators given in equation (58). We find the following electric branes

A Particle   We wish to find the electric brane associated with the representation R_0^a that has highest weight generator $R_0^{(n-1)}(00\ldots010)$, and so we set β in equation 1 to be its corresponding root. We note that $(\beta, \beta) = 1$. The lowest weight generator in the representation is $R_0^1(11\ldots110)$. To find the corresponding element in the Cartan sub-algebra we observe from Appendix B that only the root corresponding to the $(n-1)$th node does not have norm squared equal to two. Indeed $(\alpha_{n-1}, \alpha_{n-1}) = 1$, so from equation (6) we see that we have to effectively halve the H_{n-1} generator contribution when we find $\beta \cdot H$ from the root coefficients, α^i, so that

$$\beta \cdot H = H_1 + \ldots + H_{n-2} + \frac{1}{2}H_{n-1}$$

$$= \frac{D-3}{D-2}(K^1{}_1) - \frac{1}{D-2}(K^2{}_2 + \ldots + K^{n-1}{}_{n-1}) + \sqrt{\frac{2}{D-2}}R_0 \quad (64)$$

17

By substituting (64) into equation (1), and bearing in mind that $(\beta, \beta) = 1$ for R_0^1, we find the corresponding group element is

$$
\begin{aligned}
g_A &= \exp(-\ln N_0(\frac{D-3}{D-2}(K^1{}_1) - \frac{1}{D-2}(K^2{}_2 + \ldots + K^{n-1}{}_{n-1}) \\
&\quad + \sqrt{\frac{2}{D-2}}R_0)) \exp((1 - N_0)E_\beta) \\
&= \exp(-\ln N_0(\frac{D-3}{D-2}(K^1{}_1) - \frac{1}{D-2}(K^2{}_2 + \ldots + K^{n-1}{}_{n-1}))) \quad (65) \\
&\quad \exp(N_0^{-\frac{1}{D-2}}(1 - N_0)E_\beta) \exp(-\sqrt{\frac{2}{D-2}}\ln N_0 R_0)
\end{aligned}
$$

In the last line we have made use of $[R_0, R_0^{a_1}]$ from equation (59) to move the dilaton generator to the far right so that it agrees with the group element from which the non-linear realisation is constructed. By examining the R_0 term we find a dilaton given by

$$
e^A = N_0^{-\sqrt{\frac{2}{D-2}}} \tag{66}
$$

And a line element corresponding to a particle

$$
ds^2 = N_0^{-\frac{2(D-3)}{D-2}}(-dy_1^2) + N_0^{\frac{2}{D-2}}(dx_2^2 + \ldots + dx_{n-1}^2) \tag{67}
$$

The particle is derived from a gauge potential

$$
A^T{}_{1\,(0)} = N_0^{-\frac{1}{D-2}}(1 - N_0) \tag{68}
$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = N_0^{-\frac{D-3}{D-2}}$

$$
A_{1\,(0)} = N_1^{-1} - 1 \tag{69}
$$

This gives rise to a 2-form field strength, which we label $F_{\mu\nu\,0}$.

An $(n-5)$ Brane Proceeding in the usual manner we find the electric brane associated with the representation $R_1^{a_1 a_2 \ldots a_{(n-4)}}$ whose highest weight generator is $R_1^{45\ldots(n-1)}$ $(0001\ldots111)$, whose associated root we now take as β. The lowest weight generator is $R_1^{12\ldots(n-4)}$ $(1234\ldots43211)$ and the corresponding element in the Cartan sub-algebra is, recalling that the root associated with the $(n-1)$th node of the Dynkin diagram is short, being half the length of the other roots,

$$
\begin{aligned}
\beta \cdot H &= H_1 + 2H_2 + 3H_3 + 4(H_4 + \ldots + H_{n-4}) \\
&\quad + 3H_{n-3} + 2H_{n-2} + \frac{1}{2}H_{n-1} + H_n \\
&= \frac{1}{D-2}(K^1{}_1 + \ldots K^{n-4}{}_{n-4}) - \frac{D-3}{D-2}(K^{n-3}{}_{n-3} + \ldots K^{n-1}{}_{n-1}) \quad (70) \\
&\quad - \frac{1}{2}\sqrt{\frac{8}{D-2}}R_0
\end{aligned}
$$

18

We note that $(\beta, \beta) = 1$ and substitute this expression into equation (1) to find that the group element for the generator $R_1^{45\ldots(n-1)}$ is

$$g_A = \exp(-\ln N_{n-5}(\frac{1}{D-2}(K^1{}_1 + \ldots K^{n-4}{}_{n-4})$$
$$-\frac{D-3}{D-2}(K^{n-3}{}_{n-3} + \ldots K^{n-1}{}_{n-1}))) \qquad (71)$$
$$\exp(N_{n-5}^{-\frac{1}{D-2}}(1 - N_{n-5})E_\beta)$$
$$\exp(\sqrt{\frac{2}{D-2}}\ln N_{n-5}R_0)$$

We find a dilaton given by

$$e^A = N_{n-5}^{\sqrt{\frac{2}{D-2}}} \qquad (72)$$

And a line element corresponding to an $(n-5)$-brane

$$ds^2 = N_{n-5}^{-\frac{2}{D-2}}(-dy_1^2 + dx_2^2 + \ldots dx_{n-4}^2) + N_{n-5}^{\frac{2(D-3)}{D-2}}(dx_{n-3}^2 + \ldots + dx_{n-1}^2) \quad (73)$$

The particle is derived from a gauge potential

$$A^T_{12\ldots(n-4)\,(1)} = N_{n-5}^{-\frac{1}{D-2}}(1 - N_{n-5}) \qquad (74)$$

We use $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^{n-4}{}_{n-4} = N_{n-5}^{\frac{-1}{D-2}}$ to complete the change to world volume indices

$$A_{12\ldots(n-4)\,(1)} = N_{n-5}^{-1} - 1 \qquad (75)$$

This gives rise to an $(n-3)$ form field strength, which we interpret to be the dual of $F_{\mu\nu 0}$.

A String We wish to find the electric brane associated with the representation $R_0^{a_1 a_2}$ with the highest weight generator $R_0^{(n-2)(n-1)}(00\ldots0120)$, whose corresponding root we take to be β and we note that $(\beta, \beta) = 2$. The lowest weight generator is $R_0^{12}(12\ldots220)$ and the corresponding element in the Cartan sub-algebra is

$$\beta \cdot H = H_1 + 2(H_2 + \ldots + H_{n-2}) + H_{n-1}$$
$$= \frac{D-4}{D-2}(K^1{}_1 + K^2{}_2) - \frac{2}{D-2}(K^3{}_3 + \ldots K^{n-1}{}_{n-1}) + \sqrt{\frac{8}{D-2}}R_0 \quad (76)$$

The corresponding group element is

$$g_A = \exp(-\frac{1}{2}\ln N_1(\frac{D-4}{D-2}(K^1{}_1 + K^2{}_2) - \frac{2}{D-2}(K^3{}_3 + \ldots K^{n-1}{}_{n-1})))$$
$$\exp(N_1^{-\frac{2}{D-2}}(1 - N_1)E_\beta)\exp(-\sqrt{\frac{2}{D-2}}\ln N_1 R_0) \qquad (77)$$

We find a dilaton given by

$$e^A = N_0^{-\sqrt{\frac{2}{D-2}}} \qquad (78)$$

And a line element corresponding to a string

$$ds^2 = N_1^{-\frac{D-4}{D-2}}(-dy_1^2 + dx_2^2) + N_1^{\frac{2}{D-2}}(dx_3^2 + \ldots + dx_{n-1}^2) \qquad (79)$$

The string is derived from a gauge potential

$$A^T_{12\,(0)} = N_1^{-\frac{2}{D-2}}(1 - N_1) \qquad (80)$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = (e^{\hat{h}})^2{}_2 = N_1^{-\frac{D-4}{2(D-2)}}$

$$A_{12\,(0)} = N_1^{-1} - 1 \qquad (81)$$

From this we derive a 3-form field strength, which we label $F_{\mu\nu\rho 0}$.

An $(n-6)$ Brane We wish to find the electric brane associated with the representation $R_1^{a_1 a_2 \ldots a_{n-5}}$ with highest weight generator $R_1^{56\ldots(n-1)}$ $(000\ldots001)$, whose corresponding root we take as β in equation (1) and we note that $(\beta, \beta) = 2$. The associated lowest weight generator in this representation is $R_1^{12\ldots(n-5)}$ $(1234\ldots432101)$ and the corresponding element in the Cartan sub-algebra is

$$\begin{aligned}
\beta \cdot H =& H_1 + 2H_2 + 3H_3 + 4(H_4 + \ldots + H_{n-5}) \\
&+ 3H_{n-4} + 2H_{n-3} + H_{n-2} + H_n \\
=& \frac{2}{D-2}(K^1{}_1 + \ldots K^{n-5}{}_{n-5}) - \frac{D-4}{D-2}(K^{n-4}{}_{n-4} + \ldots K^{n-1}{}_{n-1}) \quad (82) \\
&- \sqrt{\frac{8}{D-2}} R_0
\end{aligned}$$

The corresponding group element is

$$\begin{aligned}
g_A = \exp(-\frac{1}{2}\ln N_{n-6}(&\frac{2}{D-2}(K^1{}_1 + \ldots K^{n-5}{}_{n-5}) \\
&- \frac{D-4}{D-2}(K^{n-4}{}_{n-4} + \ldots K^{n-1}{}_{n-1}))) \qquad (83) \\
&\exp(N_{n-6}^{-\frac{2}{D-2}}(1 - N_{n-6})E_\beta) \\
&\exp(\sqrt{\frac{2}{D-2}}\ln N_{n-6} R_0)
\end{aligned}$$

We find a dilaton given by

$$e^A = N_{n-6}^{\sqrt{\frac{2}{D-2}}} \qquad (84)$$

And a line element corresponding to an $(n-6)$-brane

$$ds^2 = N_{n-6}^{-\frac{2}{D-2}}(-dy_1^2 + dx_2^2 + \ldots dx_{n-5}^2) + N_{n-6}^{\frac{D-4}{D-2}}(dx_{n-4}^2 + \ldots + dx_{n-1}^2) \quad (85)$$

The brane is derived from a gauge potential

$$A^T_{12\ldots(n-5)\,(1)} = N_{n-6}^{-\frac{2}{D-2}}(1 - N_{n-6}) \qquad (86)$$

We use $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^{n-5}{}_{n-5} = N_{n-6}^{-\frac{1}{D-2}}$ to complete the change to world volume indices

$$A_{12\ldots(n-5)(1)} = N_{n-6}^{-1} - 1 \tag{87}$$

This gives rise to an $(n-4)$-form field strength, which we interpret to be the dual of $F_{\mu\nu\rho 0}$.

We have have successfully reproduced all the usual BPS electric branes using the group element given in equation (1). The formulae we have found above do indeed correspond to the solutions of the Lagrangian (63).

We find our dilaton field A to be related to ϕ by

$$A = -\phi \tag{88}$$

3.2   Very Extended F_4

The $F_4{}^{+++}$ Dynkin diagram is shown in Appendix B, where the solid nodes represent the gravity line with respect to which we decompose our Kac-Moody algebra. The shorter roots, enumerated as nodes 6 and 7 have $(\alpha, \alpha) = 1$ where the gravity line roots are normalised to have norm squared of two.

The F_4^{+++} algebra is decomposed with respect to its A_5 sub-algebra, the simple roots whose nodes we delete are α_7 and α_6 and their generators are $R_1(0000001)$ and $R_0^6(0000010)$. We associate the levels l_1 and l_2 with these respectively. The s labels on our generators are chosen to be the l_1 level of the generator. From reference [9] we find the $F_4{}^{+++}$ algebra decomposed with respect to an A_5 sub-algebra contains the generators $K^a{}_b$ at level (0,0) and the following generators at levels (l_1, l_2)

$l_1 \longrightarrow$		0	1	2	
$l_2 \downarrow$	0	R_0	R_1		
	1	R_0^a	R_1^a		
	2	R_0^{ab}	R_1^{ab}	$R_2{}^{ab}$	(89)
	3		R_1^{abc}	$R_2{}^{abc}$	
	4		R_1^{abcd}	$R_2{}^{abc,d}$	
				$R_2{}^{abcd}$	

The generators R_0 and $R_2{}^{abcd}$ have associated roots β such that $(\beta, \beta) = 0$ so we discard them as starting points for our method in this section, and $R_2{}^{abc,d}$, $R_2{}^{ab}$, R_0^{ab} have $(\beta, \beta) = 2$, the other generators all have $(\beta, \beta) = 1$. We make use of the following commutation relations where we have chosen the commutator coefficient for $[R_0, R_0^a]$ and the rest have been deduced from equations (10)-(13)

and the Serre relations (5)

$$[R_0, R_0^a] = \frac{1}{\sqrt{8}} R_0^a \qquad [R_0, R_1^a] = -\frac{1}{\sqrt{8}} R_1^a$$

$$[R_0, R_0^{ab}] = \frac{1}{\sqrt{2}} R_0^{ab} \qquad [R_0, R_1^{ab}] = 0 \qquad\qquad [R_0, R_2^{ab}] = -\frac{1}{\sqrt{2}} R_2^{ab}$$

$$[R_0, R_1^{abc}] = \frac{1}{\sqrt{8}} R_1^{abc} \qquad [R_0, R_2^{abc}] = -\frac{1}{\sqrt{8}} R_2^{abc}$$

$$\tag{90}$$

$$[R_0, R_1^{abcd}] = \frac{1}{\sqrt{2}} R_1^{abcd} \qquad [R_0, R_2^{abc,d}] = 0$$

The simple root generators of $F_4{}^{+++}$ are

$$E_a = K^a{}_{a+1}, a = 1, \ldots 5, E_6 = R_0^6, E_7 = R_1 \tag{91}$$

and the Cartan sub-algebra generators, H_a,is given by [10]

$$H_a = K^a{}_a - K^{a+1}{}_{a+1}, a = 1, \ldots 5,$$
$$H_6 = -\tfrac{1}{2}(K^1{}_1 + \ldots K^5{}_5) + \tfrac{3}{2}K^6{}_6 + \sqrt{2}R_0, \tag{92}$$
$$H_7 = -\sqrt{8}R_0$$

The low-level field content [9] is $\hat{h}_a{}^b$, A, $A_{a_1 0}$, $A_{a_1 1}$, $A_{a_1 a_2 2}$, $A_{a_1 a_2 a_3 a_4 1}$ and their field strengths have duals derived from $A_{a_1 a_2 a_3, b_2}$, $A_{a_1 a_2 a_3 a_4 2}$ $A_{a_1 a_2 a_3 2}$, $A_{a_1 a_2 a_3 1}$, $A_{a_1 a_2 0}$, A_1, respectively, we also have an $A_{a_1 a_2 1}$ field without a dual listed above, so we treat it as a self-dual field. Our choice of local sub-algebra for the non-linear realisation allows us to write the group element of $F_4{}^{+++}$ as

$$g = \exp(\sum_{a \leq b} \hat{h}_a{}^b K^a{}_b) \exp(\frac{1}{4!} A_{a_1 a_2 a_3 a_4 1} R_1^{a_1 a_2 a_3 a_4}) \exp(\frac{1}{4!} A_{a_1 a_2 a_3 a_4 2} R_2{}^{a_1 a_2 a_3 a_4})$$

$$\exp(\frac{1}{3!} A_{a_1 a_2 a_3, b_2} R_2{}^{a_1 a_2 a_3, b}) \exp(\frac{1}{3!} A_{a_1 a_2 a_3 2} R_2{}^{a_1 a_2 a_3})$$

$$\exp(\frac{1}{3!} A_{a_1 a_2 a_3 1} R_1^{a_1 a_2 a_3}) \exp(\frac{1}{2!} A_{a_1 a_2 2} R_2{}^{a_1 a_2}) \tag{93}$$

$$\exp(\frac{1}{2!} A_{a_1 a_2 1} R_1^{a_1 a_2}) \exp(\frac{1}{2!} A_{a_1 a_2 0} R_0^{a_1 a_2})$$

$$\exp(A_{a_1 1} R_1^a) \exp(A_{a_1 0} R_0^{a_1}) \exp(A_1 R_1) \exp(A R_0)$$

The field content of the associated maximally oxidised theory given in [11] agrees with the low-order F_4^{+++} content given above. The Lagrangian for the oxidised theory, contains two 2-form field strengths $F_{\mu\nu 0} = 2\partial_{[\mu} A_{\nu]0} + \sqrt{2} A_1 \partial_{[\mu} A_{\nu]1}$ and $F_{\mu\nu 1} = 2\partial_{[\mu} A_{\nu]1}$, a 3-form field strength $F_{\mu\nu\rho 2} = 3(\partial_{[\mu} A_{\nu\rho]2} + A_{[\mu 1} \partial_\nu A_{\rho]1})$, a self-dual 3-form field strength $F_{\mu\nu\rho 1} = 3(\partial_{[\mu} A_{\nu\rho]1} - A_{[\mu 0} \partial_\nu A_{\rho]1}) - \frac{1}{\sqrt{2}} A_1 F_{\mu\nu\rho 2}$,

a 1-form field strength $F_{\mu\,1} = \partial_\mu A_1$, and the dilaton A.

$$A = \frac{1}{16\pi G_6} \int d^6 x \sqrt{-g}(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2.2!}e^{\frac{1}{\sqrt{2}}\phi}F_{\mu\nu\,0}F_0^{\mu\nu}$$

$$- \frac{1}{2.2!}e^{-\frac{1}{\sqrt{2}}\phi}F_{\mu\nu\,1}F_1^{\mu\nu} - \frac{1}{2.3!}e^{-\sqrt{2}\phi}F_{\mu\nu\rho\,2}F_2^{\mu\nu\rho} \qquad (94)$$

$$- \frac{1}{2}e^{\sqrt{2}\phi}F_{\mu\,1}F_1^\mu)$$

$$- \int_{C.S.} d^6 x \epsilon^{a_1 \dots a_6}(\frac{1}{\sqrt{2}.3!3!}A_1 F_{a_1 a_2 a_3\,2}F_{a_4 a_5 a_6\,1} + \frac{1}{2.2!3!}A_{a_1\,0}F_{a_2 a_3\,0}F_{a_4 a_5 a_6\,2}$$

$$+ \frac{1}{2.2!3!}A_{a_1\,0}F_{a_2 a_3\,1}F_{a_4 a_5 a_6\,1})$$

The reader must remember that $F_{\mu\nu\rho\,1}$ is self-dual and its kinetic term in the action integral vanishes. We again use the group element (1) to generate all of the electric branes of F_4^{+++} starting from the generators given in equation (89). We find the following electric branes

A Particle Commencing by setting β to be the root associated with generator R_0^6 (0000010), the highest weight of the $R^{a_1}{}_0$ representation. Noting that $(\beta, \beta) = 1$ we proceed and find the lowest weight generator in this representation is R_0^1 (1111110). Taking account of the shorter root on the sixth node of the F_4^{+++} Dynkin diagram, we use equation (6) to find the element corresponding to R_0^1 in the Cartan sub-algebra

$$\beta \cdot H = (H_1 + \dots H_5) + \frac{1}{2}H_6$$

$$= \frac{3}{4}(K^1{}_1) - \frac{1}{4}(K^2{}_2 + \dots K^6{}_6) + \frac{1}{\sqrt{2}}R_0 \qquad (95)$$

And from equation (1) we find the corresponding group element is

$$g_A = \exp(- \ln N_0(\frac{3}{4}(K^1{}_1) - \frac{1}{4}(K^2{}_2 + \dots K^6{}_6) + \frac{1}{\sqrt{2}}R_0)) \exp((1 - N_0)E_\beta)$$

$$= \exp(- \ln N_0(\frac{3}{4}(K^1{}_1) - \frac{1}{4}(K^2{}_2 + \dots K^6{}_6))) \exp(N_0^{-\frac{1}{4}}(1 - N_0)E_\beta) \qquad (96)$$

$$\exp(-\frac{1}{\sqrt{2}} \ln N_0 R_0)$$

So we find a dilaton given by

$$e^A = N_0^{-\frac{1}{\sqrt{2}}} \qquad (97)$$

And a line element corresponding to a particle

$$ds^2 = N_0^{-\frac{3}{2}}(-dy_1^2) + N_0^{\frac{1}{2}}(dx_2^2 + \dots + dx_6^2) \qquad (98)$$

We have a gauge field given by

$$A^T_{1\,(0)} = N_0^{-\frac{1}{4}}(1 - N_0) \qquad (99)$$

We change to world volume indices via the vielbein $(e^h)^1{}_1 = N_0^{-\frac{3}{4}}$

$$A_{1\,(0)} = N_0^{-1} - 1 \qquad (100)$$

We associate this gauge potential with a 2-form field strength, $F_{\mu\nu\,0}$.

A 2-Brane We wish to find the electric brane associated with the representation $R^{a_1 a_2 a_3}{}_2$ whose highest weight generator is $R_2{}^{456}(0001232)$. We take this to be β in equation (1) and we note that $(\beta, \beta) = 1$. The lowest weight generator is $R_2{}^{123}(1233332)$ and the corresponding element in the Cartan sub-algebra is

$$
\begin{aligned}
\beta \cdot H &= H_1 + 2H_2 + 3(H_3 + \ldots H_5) + \frac{3}{2}H_6 + H_7 \\
&= \frac{1}{4}(K^1{}_1 + \ldots K^3{}_3) - \frac{3}{4}(K^4{}_4 + \ldots K^6{}_6) - \frac{1}{\sqrt{2}}R_0
\end{aligned}
\tag{101}
$$

Substituting this into equation (1) we find the appropriately arranged element is

$$
\begin{aligned}
g_A &= \exp(-\ln N_2(\frac{1}{4}(K^1{}_1 + \ldots K^3{}_3) - \frac{3}{4}(K^4{}_4 + \ldots K^6{}_6) - \frac{1}{\sqrt{2}}\ln N_2 R_0)) \\
&\quad \exp((1 - N_2)E_\beta) \\
&= \exp(-\ln N_2(\frac{1}{4}(K^1{}_1 + \ldots K^3{}_3) - \frac{3}{4}(K^4{}_4 + \ldots K^6{}_6) \\
&\quad \exp(N_2^{-\frac{1}{4}}(1 - N_2)E_\beta)\exp(+\frac{1}{\sqrt{2}}\ln N_2 R_0)
\end{aligned}
\tag{102}
$$

We find a dilaton given by

$$
e^A = N_2^{+\frac{1}{\sqrt{2}}}
\tag{103}
$$

And a line element corresponding to a 2-brane

$$
ds^2 = N_2^{-\frac{1}{2}}(-dy_1^2 + \ldots dx_3^2) + N_2^{\frac{3}{2}}(dx_4^2 + \ldots + dx_6^2)
\tag{104}
$$

The brane is derived from the gauge field given by

$$
A^T_{123(2)} = N_2^{-\frac{1}{4}}(1 - N_2)
\tag{105}
$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^3{}_3 = N_0^{-\frac{1}{4}}$

$$
A_{123(2)} = N_2^{-1} - 1
\tag{106}
$$

We conclude that $R_2{}^{456}$ is the generator associated with the dual of $F_{\mu\nu 0}$.

A Second Particle The calculation for the representation $R^{a_1}{}_1$ which has a highest weight generator R_1^6 is the same as that for R_0^6 except that the expansion of $\beta \cdot H$ has an extra $\frac{1}{2}H_7$ added on. That is,

$$
\beta \cdot H = \frac{3}{4}(K^1{}_1) - \frac{1}{4}(K^2{}_2 + \ldots K^6{}_6) - \frac{1}{\sqrt{2}}R_0
\tag{107}
$$

And from equation (1) we find the corresponding group element is

$$
\begin{aligned}
g_A &= \exp(-\ln N_0(\frac{3}{4}(K^1{}_1) - \frac{1}{4}(K^2{}_2 + \ldots K^6{}_6) - \frac{1}{\sqrt{2}}R_0))\exp((1 - N_0)E_\beta) \\
&= \exp(-\ln N_0(\frac{3}{4}(K^1{}_1) - \frac{1}{4}(K^2{}_2 + \ldots K^6{}_6)))\exp(N_0^{-\frac{1}{4}}(1 - N_0)E_\beta) \\
&\quad \exp(\frac{1}{\sqrt{2}}\ln N_0 R_0)
\end{aligned}
\tag{108}
$$

We find a line element identical to (98) for a particle and a dilaton given by

$$e^A = N_0^{\frac{1}{\sqrt{2}}} \tag{109}$$

We have a gauge field given by

$$A^T_{1\,(1)} = N_0^{-\frac{1}{4}}(1 - N_0) \tag{110}$$

We change to world volume indices via the vielbein $(e^h)^1{}_1 = N_0^{-\frac{3}{4}}$ and find

$$A_{1\,(1)} = N_0^{-1} - 1 \tag{111}$$

We associate this gauge potential with a 2-form field strength which we label $F_{\mu\nu\,1}$.

A Second 2-Brane   Similarly, the derivation of the electric brane associated with the highest weight generator R_1^{456} is the same as that for R_2^{456} but with a $\frac{1}{2}H_7$ taken off of expansion of the $\beta \cdot H$ expansion. That is,

$$\beta \cdot H = \frac{1}{4}(K^1{}_1 + \ldots K^3{}_3) - \frac{3}{4}(K^4{}_4 + \ldots K^6{}_6) + \frac{1}{\sqrt{2}}R_0 \tag{112}$$

Substituting this into equation (1) we find the appropriately arranged element is

$$
\begin{aligned}
g_A &= \exp(-\ln N_2(\frac{1}{4}(K^1{}_1 + \ldots K^3{}_3) - \frac{3}{4}(K^4{}_4 + \ldots K^6{}_6) + \frac{1}{\sqrt{2}}\ln N_2 R_0)) \\
&\quad \exp((1 - N_2)E_\beta) \\
&= \exp(-\ln N_2(\frac{1}{4}(K^1{}_1 + \ldots K^3{}_3) - \frac{3}{4}(K^4{}_4 + \ldots K^6{}_6) \\
&\quad \exp(N_2^{-\frac{1}{4}}(1 - N_2)E_\beta)\exp(-\frac{1}{\sqrt{2}}\ln N_2 R_0)
\end{aligned}
\tag{113}
$$

We find a dilaton given by

$$e^A = N_2^{-\frac{1}{\sqrt{2}}} \tag{114}$$

And a line element corresponding to a 2-brane identical to (104). The brane is derived from the gauge field given by

$$A^T_{123\,(1)} = N_2^{-\frac{1}{4}}(1 - N_2) \tag{115}$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^3{}_3 = N_0^{-\frac{1}{4}}$ and find

$$A_{123\,(1)} = N_2^{-1} - 1 \tag{116}$$

We conclude that R_1^{456} is the generator associated with the dual of $F_{\mu\nu\,1}$.

A String   We wish to find the electric brane associated with the $R^{a_1 a_2}{}_2$ representation whose highest weight is the generator $R_2{}^{56}(0000122)$, whose associated root we take to be β in equation (1) and we note that $(\beta, \beta) = 2$. The lowest weight generator is $R_2{}^{12}(1222222)$ and the corresponding element in the Cartan sub-algebra is

$$\beta \cdot H = H_1 + 2(H_2 + \ldots H_5) + H_6 + H_7$$
$$= \frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6) - \sqrt{2}R_0 \tag{117}$$

The corresponding group element (1) is

$$g_A = \exp(-\frac{1}{2}\ln N_1(\frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6) - \sqrt{2}\ln N_1 R_0))$$
$$\exp((1 - N_1)E_\beta)$$
$$= \exp(-\frac{1}{2}\ln N_1(\frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6))$$
$$\exp(N_1^{-\frac{1}{2}}(1 - N_1)E_\beta)\exp(\frac{1}{\sqrt{2}}\ln N_1 R_0) \tag{118}$$

We find a dilaton given by

$$e^A = N_1^{\frac{1}{\sqrt{2}}} \tag{119}$$

And a line element corresponding to a string

$$ds^2 = N_1^{-\frac{1}{2}}(-dy_1^2 + dx_2^2) + N_1^{\frac{1}{2}}(dx_3^2 + \ldots + dx_6^2) \tag{120}$$

The brane is derived from the gauge field given by

$$A^T_{12\,(2)} = N_1^{-\frac{1}{2}}(1 - N_1) \tag{121}$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = (e^{\hat{h}})^2{}_2 = N_1^{-\frac{1}{4}}$

$$A_{12\,(2)} = N_1^{-1} - 1 \tag{122}$$

This gauge potential gives rise to a 3-form field strength, which we label $F_{\mu\nu\rho\,2}$.

A Second String   Starting with with the representation $R^{a_1 a_2}{}_0$ whose highest weight is the generator $R_0^{56}(0000120)$ whose associated root we take for β in equation (1), noting that $(\beta, \beta) = 2$. The lowest weight generator in this representation is $R_0^{12}(1222220)$ which corresponds to the element in the Cartan sub-algebra given by

$$\beta \cdot H = H_1 + 2(H_2 + \ldots H_5) + H_6$$
$$= \frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6) + \sqrt{2}R_0 \tag{123}$$

The corresponding group element (1) is

$$g_A = \exp(-\frac{1}{2}\ln N_1(\frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6) + \sqrt{2}\ln N_1 R_0))$$
$$\exp((1 - N_1)E_\beta)$$
$$= \exp(-\frac{1}{2}\ln N_1(\frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6))$$
$$\exp(N_1^{-\frac{1}{2}}(1 - N_1)E_\beta)\exp(-\frac{1}{\sqrt{2}}\ln N_1 R_0) \tag{124}$$

We find a dilaton given by

$$e^A = N_1^{-\frac{1}{\sqrt{2}}} \tag{125}$$

And a line element corresponding to a string

$$ds^2 = N_1^{-\frac{1}{2}}(-dy_1^2 + dx_2^2) + N_1^{\frac{1}{2}}(dx_3^2 + \ldots + dx_6^2) \tag{126}$$

The brane is derived from the gauge field given by

$$A_{12(0)}^T = N_1^{-\frac{1}{2}}(1 - N_1) \tag{127}$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = (e^{\hat{h}})^2{}_2 = N_1^{-\frac{1}{4}}$

$$A_{12(0)} = N_1^{-1} - 1 \tag{128}$$

We conclude that this is the group element associated with the dual of $F_{\mu\nu\rho_2}$.

A Gooseberry String   We find the electric brane associated with the representation $R^{a_1 a_2}{}_1$ with highest weight generator $R_1^{56}(0000121)$, whose associated root we take for β in equation (1), noting that $(\beta, \beta) = 1$. The lowest weight generator in this representation is $R_1^{12}(1222221)$ and this corresponds to the Cartan sub-algebra element

$$\beta \cdot H = H_1 + 2(H_2 + \ldots H_5) + H_6 + \frac{1}{2}H_7$$
$$= \frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6) \tag{129}$$

Substituting this expression into equation (1) gives the corresponding element

$$g_A = \exp(-\ln N_1(\frac{1}{2}(K^1{}_1 + K^2{}_2) - \frac{1}{2}(K^3{}_3 + \ldots K^6{}_6))) \exp((1 - N_1)E_\beta) \tag{130}$$

There is no dilaton and we find a line element corresponding to a string

$$ds^2 = N_1^{-1}(-dy_1^2 + dx_2^2) + N_1(dx_3^2 + \ldots + dx_6^2) \tag{131}$$

The brane is derived from the gauge field given by

$$A_{12(1)}^T = (1 - N_1) \tag{132}$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = (e^{\hat{h}})^2{}_2 = N_1^{-\frac{1}{2}}$

$$A_{12(1)} = N_1^{-1} - 1 \tag{133}$$

This gauge field is associated with a second 3-form field strength which we label, $F_{\mu\nu\rho_1}$. We deduce that this field strength is self-dual as we have no more two-form generators in (89) that could produce an electric brane that would be associated with a dual to $F_{\mu\nu\rho_1}$. We note that the difference in the line element for the string here and those for the other strings derived from F_4^{+++} is due to the dilaton coupling constant being zero here.

27

A 3-Brane   The representation $R_1^{a_1 a_2 a_3 a_4}$ has highest weight generator R_1^{3456} (0012341) whose associated root we take as β in equation (1), we note that $(\beta, \beta) = 1$. The lowest weight generator in this representation is R_1^{1234}(1234441). This corresponds to an element in the Cartan sub-algebra given by

$$\beta \cdot H = H_1 + 2H_2 + 3H_3 + 4(H_4 + H_5) + 2H_6 + \frac{1}{2}H_7$$

$$= 0(K^1{}_1 + \ldots K^4{}_4) - (K^5{}_5 + K^6{}_6) + \sqrt{2}R_0 \tag{134}$$

The group element given by equation (1) is

$$g_A = \exp(-\ln N_3(0(K^1{}_1 + \ldots K^4{}_4) - (K^5{}_5 + K^6{}_6) + \sqrt{2}R_0))\exp((1 - N_3)E_\beta)$$

$$= \exp(-\ln N_3(0(K^1{}_1 + \ldots K^4{}_4) - (K^5{}_5 + K^6{}_6)))\exp(N_3^{-1}(1 - N_3)E_\beta) \tag{135}$$

$$\exp(-\sqrt{2}\ln N_3 R_0)$$

We find a dilaton given by

$$e^A = N_3^{-\sqrt{2}} \tag{136}$$

And a line element corresponding to a 3-brane

$$ds^2 = (-dy_1^2 + \ldots dx_4^2) + N_3^2(dx_5^2 + dx_6^2) \tag{137}$$

The associated gauge potential is

$$A_{1234}{}_{(1)}^T = N_3^{-1} - 1 \tag{138}$$

The expression is the same in world volume indices as $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^4{}_4 = 1$ that is

$$A_{1234(1)} = N_3^{-1} - 1 \tag{139}$$

This gauge potential gives rise to a 5 form field strength and we conclude this is the dual to the one form field strength formed from R_1, $F_{\mu 1} = \partial_\mu R_1$.

We have have successfully reproduced all the usual BPS electric branes using the group element given in equation (1). The formulae we have found above do indeed correspond to the solutions of the Lagrangian (95).

We find our dilaton field A to be related to ϕ by

$$A = -\phi \tag{140}$$

4   Higher Level Branes

In this paper we generalised the result of [3], namely that the group element of equation (1) encodes the usual electric BPS brane solutions of the $\mathcal{G}^{+++}$ theories at low levels. So far we have only considered the low-order field content, up to the level of the dual gravity field. We have found precise agreement with the solutions of the known actions of the oxidised theories [11]. The *pp*-wave for each $\mathcal{G}^{+++}$ theory is also present in the low order theory and can be found as advocated in [3]. However, as explained in [3] we can also apply equation (1) to

the higher order generators of any $\mathcal{G}^{+++}$ to find putative solutions solutions to the non-linearly realised theory.[4] In this paper we do not show that these are actually solutions but the elegance of equation (1) leads us to hope that they are.

Let us consider the example for the F_4^{+++} theory. Field content at higher levels in F_4^{+++} is given in [9], and we find the field $A_{a_1 a_2 a_3 a_4\,3}$ at level $(3, 4)$. This representation has a highest weight generator R_3^{3456} (0012343) and commutes with the dilaton via $[R_0, R_3^{a_1 a_2 a_3 a_4}] = -\frac{1}{\sqrt{2}} R_3^{a_1 a_2 a_3 a_4}$. Applying our method for the root β associated with this generator, for which we note that $(\beta, \beta) = 1$, we find a group element from (1) given by

$$g_A = \exp(-\ln N_3(0(K^1{}_1 + \ldots K^4{}_4) - (K^5{}_5 + K^6{}_6))) \tag{141}$$
$$\exp(N_3^{-1}(1 - N_3)E_\beta) \exp(\sqrt{2} \ln N_3 R_0)$$

A dilaton given by

$$e^A = N_3^{\sqrt{2}} \tag{142}$$

A line element corresponding to a 3-brane

$$ds^2 = (-dy_1^2 + \ldots dx_4^2) + N_3(dx_5^2 + dx_6^2) \tag{143}$$

The brane is derived from a gauge potential

$$A_{1234\,(3)}^T = N_3^{-1} - 1 = A_{1234\,(3)} \tag{144}$$

This gives rise to a 5-form field strength, $F_{\mu\nu\rho\sigma\tau\,3}$. If the dual field strength were to exist we would expect it to be a 1-form formed from a scalar, which we label R_{-1}. Such a scalar does not appear in the field content tables of [9] and does not exist but had it done it would commute with the dilaton via $[R_0, R_{-1}] = \frac{1}{\sqrt{8}} R_{-1}$.

As is well known, the eleven dimensional supergravity theory has a *pp*-wave, a 2-brane and a 5-brane whose corresponding conserved charges occur in the eleven dimensional supersymmetry algebra. This is consistent with the results of [20] which showed that each of the brane solutions had a topological charge that appeared as a central charge in the supersymmetry algebra. In reference [22] it was argued that the brane solutions of the non-linearly realised $\mathcal{G}^{+++}$ theory possessed charges which belonged to the l_1 representation of $\mathcal{G}^{+++}$. The l_1 representation is the fundamental representation of $\mathcal{G}^{+++}$ associated with the very-extended node on the corresponding Dynkin diagram. The l_1 multiplet of charges for E_8^{+++} begins with the generators of spacetime translations, P^a, and then contains the 2-form and the 5-form central charges of the supersymmetry algebra at the lowest levels, as well as an infinite number of other charges in the higher orders. It was shown in [21] that the l_1 representation has the correct A_n representation to be interpreted as central charges. Indeed the reader may verify this correspondence for the brane solutions found in this paper for each $\mathcal{G}^{+++}$ considered. Let us consider the example of F_4^{+++} and list the fields, $A_{[n]}$, with

---

[4]Actually in reference [3] one should have used the formula (1) given here to investigate the higher order brane solutions, as the root β associated with any higher order generator will generally not satisfy $(\beta, \beta) = 2$. Fortunately, for all but the brane associated with the generator $R_{b_1 b_2}^{a_1 a_2 a_3}$, the higher order solutions considered in that paper have $(\beta, \beta) = 2$.

their corresponding conserved charges, $Z^{[n-1]}$, associated with the brane solution we have found in each case, including the higher level field, $A_{a_1a_2a_3a_4\,(s=3)}$, which we have just considered

Levels, (l_1, l_2)	Fields, $A_{[n]}$	Associated Charges, $Z^{[n-1]}$
$(0,0)$	$\hat{h}_a{}^b$	P^a
$(0,0)$	A	$-$
$(1,0)$	A_1	$-$
$(0,1),(1,1)$	$A^i_{a_1}$	Z^i
$(0,2),(1,2),(2,2)$	$A^{(ij)}_{a_1a_2}$	$Z^{a_1(ij)}$
$(1,3),(2,3)$	$A^i_{a_1a_2a_3}$	$Z^{a_1a_2i}$
$(2,4)$	$A_{a_1a_2a_3,b}$	$Z^{a_1a_2,b}$
$(1,4),(2,4),(3,4)$	$A^{(ij)}_{a_1a_2a_3a_4}$	$Z^{a_1a_2a_3(ij)}$

The F_4^{+++} theory possesses an $SL(2)$ symmetry associated with the seventh node of its Dynkin diagram, which has the generator R_1, which gives rise to the assignment of the i,j indices above. We note that while we could not use our method to find the brane solution associated to $A_{a_1a_2a_3a_4\,(s=2)}$, as its generator had associated root, β, such that $(\beta, \beta) = 0$, we are able to deduce that its field strength is dual to a 1-form field strength formed from the dilaton, A. Indeed we expect its brane solution to have a third rank tensor conserved charge, as listed above.

It is interesting to compare these with the supersymmetric central charges of the $(1,0)$ supersymmetry algebra

$$\left\{ Q^i_\alpha, Q^j_\beta \right\} = \epsilon^{ij}(\Gamma_m)_{\alpha\beta}P^m + (\Gamma_{m_1m_2m_3})_{\alpha\beta}Z^{m_1m_2m_3(ij)} \qquad (145)$$

The supersymmetric generator of spacetime translations, P^m, is equivalent to the conserved charge arising from the pp-wave solution corresponding to the graviton, $\hat{h}_a{}^b$, in the F_4^{+++} field content. Similarly we can make an association between the central charge $Z^{m_1m_2m_3(ij)}$ of the supersymmetry algebra and the conserved charges coming from brane solutions which couple to the fields $A^i_{a_1a_2a_3a_4}$. The 5-form field strengths constructed from $A_{a_1a_2a_3a_4\,(s=1)}$ and $A_{a_1a_2a_3a_4\,(s=2)}$ are dual to the 1-form field strengths formed from the axion, A_1, and the dilaton, A, respectively, which can be seen by referring to the 3-brane solution found on page 28 and from considering the low-level content. We have just considered, above in this section, the field, $A_{a_1a_2a_3a_4\,(s=3)}$, whose field strength is a 5-form and its associated brane solution is also a 3-brane. Altogether we have a triplet of conserved charges coming from the non-linear realisation of F_4^{+++} which are equivalent to the central charges, $Z^{m_1m_2m_3(ij)}$, of the supersymmetry algebra. Including the generators of spacetime translations, P^m, the conserved charges, $Z^{m_1m_2m_3(ij)}$, of F_4^{+++} account for 36 degrees of freedom which is compatible with the anticommutator of the two $Sp(2)$ Majorana-Weyl spinors, Q^i_α.

In this paper we have constructed branes, even at a lower level than those mentioned above, whose conserved charges in the above table are not present amongst the supersymmetry algebra's central charges; charges such as Z^i and

the part of $Z^{a_1(ij)}$ triplet associated with the so-called "gooseberry string", listed in our analysis of F_4^{+++} on page 27. Indeed we can carry on and consider the brane solutions associated with the infinite number of higher-order fields in F_4^{+++}, or any $\mathcal{G}^{+++}$, and find a conserved charge for each one. We therefore find that the central charges in the supersymmetry algebra only account for a few of the brane charges, all of which belong to the l_1 representation.

Acknowledgements

This paper extends the considerations of [3] to find a universal formula for the $\mathcal{G}^{+++}$ elements which expresses the usual BPS solutions for single branes found in $\mathcal{G}^{+++}$ at low orders. While this paper was in the final stages of preparation a paper with related results [23] was submitted to the archive.

A The Electric Branes of E_6^{+++}, E_7^{+++} and G_2^{+++}

A.1 Very Extended E_6

The Dynkin diagram for E_6^{+++} is given in Appendix B. The E_6^{+++} algebra may only be uniquely decomposed with respect to an A_7 gravity line, giving an 8-dimensional theory. The simple roots that we eliminate are α_9 and α_8 whose corresponding generators are $R_1(00\ldots001)$ and $R_0^{678}(00\ldots010)$ and we associate the levels l_1 and l_2 with these respectively. The s labels on our generators are chosen to be the l_1 level of the generator. The alternative decomposition along the other branch of the Dynkin diagram is related to the first decomposition via an automorphism.

From reference [9] we find the remaining algebra contains the generators $K^a{}_b$ at level $(0,0)$ and the following generators at up to level $(1,2)$

$$
\begin{array}{cccc}
l_1 \longrightarrow & & 0 & 1 \\
l_2 \downarrow & 0 & R_0 & R_1 \\
 & 1 & R_0^{a_1 a_2 a_3} & R_1^{a_1 a_2 a_3} \\
 & 2 & R_0^{a_1 \ldots a_6} & R_1^{a_1 \ldots a_6} \\
 & & & R_1^{a_1 \ldots a_5, b}
\end{array}
\tag{146}
$$

All the generators have $(\beta, \beta) = 2$, where β is the associated root for the generator, with the exceptions of R_0 and $R_1^{a_1 \ldots a_6}$ which have $(\beta, \beta) = 0$ and so will not be used as starting points for finding electric branes.

We make use of the following commutator relations where we have chosen the commutator coefficient for $[R_0, R_1]$ and the others have followed from equations (10)-(13) and the Serre relations (5)

$$[R_0, R_1] = -R_1$$

$$[R_0, R_0^{a_1 a_2 a_3}] = \frac{1}{2} R_0^{a_1 a_2 a_3} \qquad [R_0, R_1^{a_1 a_2 a_3}] = -\frac{1}{2} R_1{}^{a_1 a_2 a_3} \tag{147}$$

$$[R_0, R_0^{a_1 \ldots a_6}] = R_0^{a_1 \ldots a_6} \qquad [R_0, R_1^{a_1 \ldots a_6}] = 0$$

The simple root generators of E_6^{+++} are

$$E_a = K^a{}_{a+1}, a = 1, \ldots 7, E_8 = R_0^{678}, E_9 = R_1 \tag{148}$$

and the Cartan sub-algebra generators, H_a, are given by

$$H_a = K^a{}_a - K^{a+1}{}_{a+1}, a = 1, \ldots 7,$$
$$H_8 = -\tfrac{1}{2}(K^1{}_1 + \ldots K^5{}_5) + \tfrac{1}{2}(K^6{}_6 + \ldots K^8{}_8) + R_0, \tag{149}$$
$$H_9 = -2R_0$$

The low-level field content [9] is $\hat{h}_a{}^b$, A, $A_{a_1 a_2 a_3 0}$, $A_{a_1 \ldots a_6 0}$ and their field strengths have duals derived from $A_{a_1 \ldots a_5, b_1}$, $A_{a_1 \ldots a_6 1}$, $A_{a_1 a_2 a_3 1}$, A_1, respectively. Our choice of local sub-algebra for the non-linear realisation allows us to

write the group element of E_6^{+++} as

$$g = \exp(\sum_{a \leq b} \hat{h}_a{}^b K^a{}_b) \exp(\frac{1}{6!} A_{a_1 \ldots a_6\,1} R_1^{a_1 \ldots a_6}) \exp(\frac{1}{6!} A_{a_1 \ldots a_6\,0} R_0^{a_1 \ldots a_6})$$

$$\exp(\frac{1}{5!} A_{a_1 \ldots a_5, b_1} R_1^{a_1 \ldots a_5, b}) \exp(\frac{1}{3!} A_{a_1 a_2 a_3\,1} R_1^{a_1 \ldots a_3}) \quad (150)$$

$$\exp(\frac{1}{3!} A_{a_1 a_2 a_3\,0} R_0^{a_1 \ldots a_3}) \exp(A_1 R_1) \exp(A R_0)$$

The field content of the associated maximally oxidised theory given in [11] agrees with the low-order E_6^{+++} content given above. The Lagrangian for the oxidised theory, contains contains a graviton, ϕ, a 4-form field strength, $F_{\mu\nu\rho\sigma\,0} = 4\partial_{[\mu} A_{\nu\rho\sigma]\,0}$, a 1-form field strength, $F_{\mu\,1} = \partial_\mu A_1$ and a dilaton A and is given by

$$A = \frac{1}{16\pi G_{n-1}} \int d^8 x \sqrt{-g} (R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{2\phi} F_{\mu\,1} F_1^\mu$$

$$- \frac{1}{2.4!} e^{-\phi} F_{\mu\nu\rho\sigma\,0} F_0^{\mu\nu\rho\sigma}) \quad (151)$$

$$+ \int_{C.S.} d^8 x \epsilon^{a_1 \ldots a_8} \frac{1}{4!4!} A_1 F_{a_1 \ldots a_4\,0} F_{a_5 \ldots a_8\,0}$$

Using the group element (1) we find the following electric branes

A 2-Brane Let us find the electric brane associated with the representation $R^{a_1 a_2 a_3}{}_0$ whose highest weight generator is R_0^{678} $(00 \ldots 010)$, whose associated root we take as β in equation (1). The lowest weight generator in this representation is R_0^{123} $(123 \ldots 32110)$ and the associated Cartan sub-algebra element is

$$\beta \cdot H = H_1 + 2H_2 + 3(H_3 + \ldots H_5) + 2H_6 + H_7 + H_8$$

$$= \frac{1}{2}(K^1{}_1 + \ldots K^3{}_3) - \frac{1}{2}(K^4{}_4 + \ldots K^8{}_8) + R_0 \quad (152)$$

Bearing in mind that $(\beta, \beta) = 2$, we find that the group element (1) is

$$g_A = \exp(-\frac{1}{2} \ln N_2 (\frac{1}{2}(K^1{}_1 + \ldots K^3{}_3) - \frac{1}{2}(K^4{}_4 + \ldots K^8{}_8)))$$

$$\exp(N_2^{-\frac{1}{4}}(1 - N_2) E_\beta) \exp(-\frac{1}{2} \ln N_2 R_0) \quad (153)$$

We have a dilaton given by

$$e^A = N_2^{-\frac{1}{2}} \quad (154)$$

And a line element corresponding to a 2-brane

$$ds^2 = N_2^{-\frac{1}{2}}(-dy_1^2 + dx_2^2 + dx_3^2) + N_2^{\frac{1}{2}}(dx_4^2 + \ldots + dx_8^2) \quad (155)$$

The brane is derived from a gauge potential

$$A_{123\,(0)}^T = N_2^{-\frac{1}{4}}(1 - N_2) \quad (156)$$

The change to world volume indices is made using $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^3{}_3 = N_2^{-\frac{1}{4}}$

$$A_{123\,(0)} = N_2^{-1} - 1 \quad (157)$$

This gauge potential gives rise to a 4-form field strength, $F_{\mu\nu\rho\sigma\,0}$.

A Second 2-Brane   Let us find the brane associated with the representation $R^{a_1}{}_1$ whose highest weight generator is R_1^{678} $(00\ldots011)$, whose associated root we take for β in equation (1). The lowest weight generator in this representation is R_1^{123} $(123\ldots32111)$ and the corresponding element in the Cartan sub-algebra is identical to that for R_0^{123} but with $2R_0$ taken off, that is,

$$\beta \cdot H = \frac{1}{2}(K^1{}_1 + \ldots K^3{}_3) - \frac{1}{2}(K^4{}_4 + \ldots K^8{}_8) - R_0 \tag{158}$$

We find that the group element of equation (1) is

$$
\begin{aligned}
g_A = {} & \exp(-\frac{1}{2}\ln N_2(\frac{1}{2}(K^1{}_1 + \ldots K^3{}_3) - \frac{1}{2}(K^4{}_4 + \ldots K^8{}_8))) \\
& \exp(N_2^{-\frac{1}{4}}(1 - N_2)E_\beta)\exp(\frac{1}{2}\ln N_2 R_0)
\end{aligned}
\tag{159}
$$

We have a dilaton given by

$$e^A = N_2^{\frac{1}{2}} \tag{160}$$

And a line element corresponding to a 2-brane

$$ds^2 = N_2^{-\frac{1}{2}}(-dy_1^2 + dx_2^2 + dx_3^2) + N_2^{\frac{1}{2}}(dx_4^2 + \ldots + dx_8^2) \tag{161}$$

The brane is derived from a gauge potential

$$A^T_{123(1)} = N_2^{-\frac{1}{4}}(1 - N_2) \tag{162}$$

The change to world volume indices is made using $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^3{}_3 = N_2^{-\frac{1}{4}}$

$$A_{123(1)} = N_2^{-1} - 1 \tag{163}$$

This gives rise to a 4-form field strength, which we conclude is the dual to $F_{\mu\nu\rho\sigma\,0}$.

A 5-Brane   Let us find the electric brane associated with the representation $R^{a_1 a_2 \ldots a_6}{}_0$ which has highest weight generator $R_0^{345678}(001232120)$, whose associated root we take as β in equation (1). The lowest weight generator in this representation is $R_0^{123456}(123454220)$ and the corresponding element in the Cartan sub-algebra is

$$
\begin{aligned}
\beta \cdot H = {} & H_1 + 2H_2 + 3H_3 + 4H_4 + 5H_5 + 4H_6 + 2H_7 + 2H_8 \\
= {} & 0(K^1{}_1 + \ldots K^6{}_6) - (K^7{}_7 + K^8{}_8) + 2R_0
\end{aligned}
\tag{164}
$$

The group element (1) is

$$
\begin{aligned}
g_A = {} & \exp(\frac{1}{2}\ln N_5(K^7{}_7 + K^8{}_8)) \\
& \exp(N_5^{-1}(1 - N_5)E_\beta)\exp(-\ln N_5 R_0)
\end{aligned}
\tag{165}
$$

We have a dilaton given by

$$e^A = N_5^{-1} \tag{166}$$

And a line element corresponding to a 5-brane

$$ds^2 = (-dy_1^2 + dx_2^2 + \ldots dx_6^2) + N_5(dx_7^2 + dx_8^2) \tag{167}$$

The brane is derived from a gauge potential

$$A^T_{12\ldots6\,(0)} = N_5^{-1} - 1 \tag{168}$$

The change to world volume indices leaves the form of the gauge potential unaltered as $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^6{}_6 = 1$

$$A_{12\ldots6\,(0)} = N_5^{-1} - 1 \tag{169}$$

This gauge potential gives rise to a 7-form field strength, which we conclude is the dual to the 1 form $F_{\mu 1} = \partial_\mu R_1$.

We have have successfully reproduced all the usual BPS electric branes using the group element given in equation (1). The formulae we have found above do indeed correspond to the solutions of the Lagrangian (152).

We find our dilaton field A to be related to ϕ by

$$A = \phi \tag{170}$$

A.2 Very Extended E_7 - The 10-dimensional theory

The E_7^{+++} algebra may be decomposed with respect to an A_9 gravity line, giving a 10-dimensional theory or equally with respect to its A_7 gravity line to give an 8-dimensional theory. We consider the 10-dimensional theory here, and note that in doing so we depart from the considerations of [11], in which the 9-dimensional theory is considered.

The simple root whose node we delete is α_{10} and has generator R^{78910} $(00\ldots001)$ and we associate the level l with it. Our s labels are all zero, as we eliminate only one simple root, and are discarded in this section. From reference [9] we find the E_7^{+++} algebra decomposed with respect to an A_9 sub-algebra contains the generators $K^a{}_b$ at level 0 and the following generators up to level 2, and notably no dilaton generator,

$$
\begin{array}{cl}
l \downarrow & \\
1 & R^{a_1 a_2 a_3 a_4} \\
2 & R^{a_1 \ldots a_7, b} \\
& R^{a_1 \ldots a_8}
\end{array}
\tag{171}
$$

We note that the generator $R^{a_1 \ldots a_8}$ has associated root β such that $(\beta, \beta) = 0$ so we discard this as a starting point for finding an electric brane, all the other generators have $(\beta, \beta) = 2$. The simple root generators of E_7^{+++} are

$$E_a = K^a{}_{a+1}, a = 1, \ldots 9, E_{10} = R^{78910} \tag{172}$$

and the Cartan sub-algebra generators, H_a, are given by

$$H_a = K^a{}_a - K^{a+1}{}_{a+1}, a = 1, \ldots 9,$$
$$H_{10} = -\frac{1}{2}(K^1{}_1 + \ldots K^6{}_6) + \frac{1}{2}(K^7{}_7 + \ldots K^{10}{}_{10}) \tag{173}$$

The low-level field content [9] is $\hat{h}_a{}^b$, and its field strength has a dual derived from $A_{a_1 \ldots a_7, b}$, we also have fields $A_{a_1 a_2 a_3 a_4}$, $A_{a_1 \ldots a_8}$ which are not related to each other by a duality condition and we take them to be self-dual in our low order theory. Our choice of local sub-algebra for the non-linear realisation allows us to write the group element of E_7^{+++} as

$$g = \exp(\sum_{a \leq b} \hat{h}_a{}^b K^a{}_b) \exp(\frac{1}{8!} A_{a_1 \ldots a_8} R^{a_1 \ldots a_8}) \exp(\frac{1}{7!} A_{a_1 \ldots a_7, b} R^{a_1 \ldots a_7, b})$$

$$\exp(\frac{1}{4!} A_{a_1 \ldots a_4} R^{a_1 \ldots a_4}) \tag{174}$$

A.2.1 A 3-Brane

Let us find the electric brane associated with the representation $R^{a_1 a_2 a_3 a_4}$ whose highest weight generator is R^{78910} $(00 \ldots 001)$, whose root, β, we use in equation (1). The lowest weight generator in this representation is R^{1234} (1234443211) which corresponds to the Cartan sub-algebra element given by

$$\beta \cdot H = H_1 + 2H_2 + 3H_3 + 4(H_4 + \ldots H_6) + 3H_7 + 2H_8 + H_9 + H_{10}$$

$$= \frac{1}{2}(K^1{}_1 + \ldots K^4{}_4) - \frac{1}{2}(K^5{}_5 + \ldots K^{10}{}_{10}) \tag{175}$$

We now write down the group element from equation (1)

$$g_A = \exp(-\frac{1}{2} \ln N_3(\frac{1}{2}(K^1{}_1 + \ldots K^4{}_4) - \frac{1}{2}(K^5{}_5 + \ldots K^{10}{}_{10}))) \exp((1 - N_3)E_\beta) \tag{176}$$

We have a line element corresponding to a 3-brane

$$ds^2 = N_3^{-\frac{1}{2}}(-dy_1^2 + dx_2^2 + \ldots dx_4^2) + N_3^{\frac{1}{2}}(dx_5^2 + \ldots + dx_{10}^2) \tag{177}$$

The brane is derived from a gauge potential

$$A^T_{1234} = 1 - N_3 \tag{178}$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = \ldots (e^{\hat{h}})^4{}_4 = N_3^{-\frac{1}{4}}$

$$A_{1234} = N_3^{-1} - 1 \tag{179}$$

This gives rise to a 5-form field strength, $F_{\mu\nu\rho\sigma\tau}$ which we conclude is self-dual, as there are no other other generators in (171) from which we could derive a dual field strength, and consequently we cannot construct a Lagrangian for this theory.

A.3 Very Extended G_2

The Dynkin diagram for $G_2{}^{+++}$ is shown in Appendix B, with the darkened nodes indicating the gravity line we consider here. The $G_2{}^{+++}$ algebra is decomposed with respect to the A_4 sub-algebra of the gravity line where the deleted node corresponds to simple root is α_5 whose generator is $R^5(00001)$, to which our decomposition level, l, is associated. Again we have no need for the s labels as they are all zero, so we discard them in this section. Reference [9] gives the generators at low levels, we find the $K^a{}_b$ at the zeroth level corresponding to the gravity line step operators and we have the following other generators up to level 3, and notably no dilaton generator,

$$
\begin{array}{cc}
l \downarrow & \\
1 & R^a \\
2 & R^{ab} \\
3 & R^{ab,c} \\
& R^{abc}
\end{array}
\tag{180}
$$

The generators R^a, R^{ab} and $R^{ab,c}$ each have an associated root β such that $(\beta,\beta) = \frac{2}{3}$ but R^{abc} has $(\beta,\beta) = 0$ so we shall discard it as a starting point for our method of finding electric branes.

The simple root generators of $G_2{}^{+++}$ are

$$
E_a = K^a{}_{a+1}, a = 1, \dots 4, E_5 = R^5
\tag{181}
$$

and the Cartan sub-algebra generators, H_a, are given by

$$
\begin{aligned}
H_a &= K^a{}_a - K^{a+1}{}_{a+1}, a = 1, \dots 4, \\
H_5 &= -(K^1{}_1 + \dots K^4{}_4) + 2K^5{}_5
\end{aligned}
\tag{182}
$$

The low-level field content [9] is $\hat{h}_a{}^b$, A_{a_1} and their field strengths have duals derived from $A_{a_1a_2,b}$, $A_{a_1a_2}$, respectively. We also find the field $A_{a_1a_2a_3}$ which is not related to any of the other fields by a duality condition and we take it to be self-dual in our low order theory. Our choice of local sub-algebra for the non-linear realisation allows us to write the group element of G_2^{+++} as

$$
g = \exp(\sum_{a \leq b} \hat{h}_a{}^b K^a{}_b) \exp(\frac{1}{3!} A_{a_1a_2a_3} R^{a_1a_2a_3}) \exp(\frac{1}{2!} A_{a_1a_2,b} R^{a_1a_2,b})
\tag{183}
$$

$$
\exp(\frac{1}{2!} A_{a_1a_2} R^{a_1a_2}) \exp(A_{a_1} R^{a_1})
$$

The field content of the associated maximally oxidised theory given in [11] agrees with the low-order G_2^{+++} content given above. The Lagrangian for the oxidised theory, contains contains a graviton, ϕ, and contains a 2-form field strength $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$, and is given by

$$
A = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}(R - \frac{1}{2.2!} F_{\mu\nu} F^{\mu\nu}) + \int_{C.S.} d^5x \epsilon^{a_1 \dots a_5} \frac{1}{3!2!\sqrt{3}} F_{a_1a_2} F_{a_3a_4} A_{a_5}
\tag{184}
$$

We find the following electric branes of G_2^{+++} via our group element (1)

A Particle We find the highest weight generator associated with the representation R^{a_1} is $R^5(00001)$, whose associated root we take for β in equation (1), noting that $(\beta, \beta) = \frac{2}{3}$. We find that the lowest weight generator in this representation is $R^1(11111)$. Taking account that the fifth simple root, as enumerated in Appendix B, of G_2^{+++} is shorter than the other four such that $\frac{(\alpha_a, \alpha_a)}{(\alpha_5, \alpha_5)} = 3$ where $a = 1 \ldots 4$ we find that the expansion of R^1 in the Cartan sub-algebra according to (6) is

$$
\begin{aligned}
\beta \cdot H &= H_1 + H_2 + H_3 + H_4 + \frac{1}{3} H_5 \\
&= \frac{2}{3}(K^1{}_1) - \frac{1}{3}(K^2{}_2 + \ldots K^5{}_5)
\end{aligned}
\tag{185}
$$

The group element in equation (1) takes the form

$$
g_A = \exp\left(-\frac{3}{2} \ln N_0 \left(\frac{2}{3}(K^1{}_1) - \frac{1}{3}(K^2{}_2 + \ldots K^5{}_5)\right)\right) \exp((1 - N_0) E_\beta) \tag{186}
$$

We read off the line element of a particle

$$
ds^2 = N_0^{-2}(-dy_1^2) + N_0(dx_2^2 + \ldots + dx_5^2) \tag{187}
$$

The associated gauge potential is read from the group element to be

$$
A_1{}^T = (1 - N_0) \tag{188}
$$

We complete the change to world volume indices using $(e^{\hat{h}})^1{}_1 = N_0^{-1}$

$$
A_1 = N_0^{-1} - 1 \tag{189}
$$

This gauge potential gives rise to a 2-form field strength, $F_{\mu\nu}$.

A String The highest weight generator in the representation $R^{a_1 a_2}$ is R^{45} (00012) whose associated root β we take for β in equation (1). The lowest weight generator in this representation is $R^{12}(12222)$ and, noting that $(\beta, \beta) = \frac{2}{3}$, we find that the corresponding element in the Cartan sub-algebra is

$$
\begin{aligned}
\beta \cdot H &= H_1 + 2(H_2 + \ldots H_4) + \frac{2}{3} H_5 \\
&= \frac{1}{3}(K^1{}_1 + K^2{}_2) - \frac{2}{3}(K^3{}_3 + \ldots K^5{}_5)
\end{aligned}
\tag{190}
$$

The group element of equation (1) takes the form

$$
g_A = \exp\left(-\frac{3}{2} \ln N_1 \left(\frac{1}{3}(K^1{}_1 + K^2{}_2) - \frac{2}{3}(K^3{}_3 + \ldots K^5{}_5)\right)\right) \exp((1 - N_1) E_\beta) \tag{191}
$$

We have the line element of a string

$$
ds^2 = N_1^{-1}(-dy_1^2 + dx_2^2) + N_1^2(dx_3^2 + \ldots + dx_5^2) \tag{192}
$$

The form of the line element is the same as that of the brane associated with the dual of $F_{\mu\nu}$, the 2-form field strength which we obtained in the previous section. The associated gauge potential is read from the group element to be
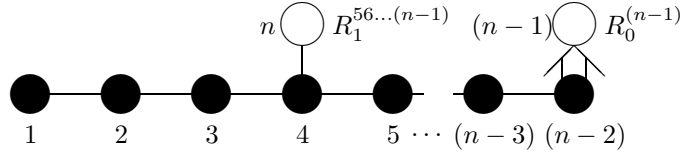
$$
A_{12}{}^T = (1 - N_1) \tag{193}
$$

We change to world volume indices using $(e^{\hat{h}})^1{}_1 = (e^{\hat{h}})^2{}_2 = N_1^{-\frac{1}{2}}$

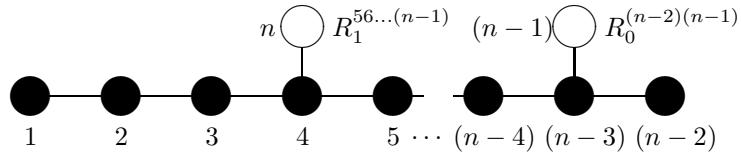$$A_{12} = N_1^{-1} - 1 \tag{194}$$

We have have successfully reproduced all the usual BPS electric branes of the oxidised G_2 theory using the group element given in equation (1). The formulae we have found above do indeed correspond to the solutions of the Lagrangian (184).
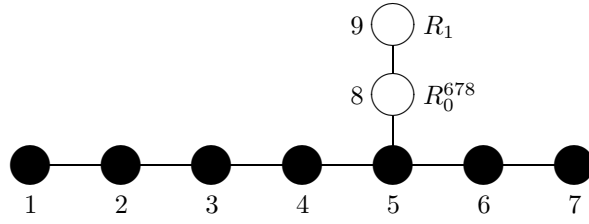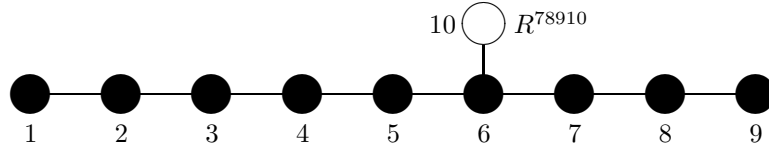
B Dynkin Diagrams

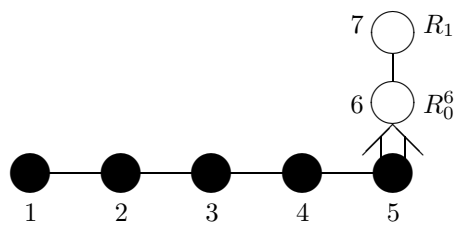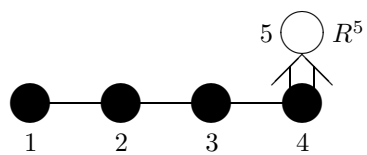## B.1 B_{n-3}^{+++}

B.2 D_{n-3}^{+++}

B.3 E_6^{+++}

B.4 E_7^{+++}

B.5 F_4^{+++}

B.6 G_2^{+++}

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